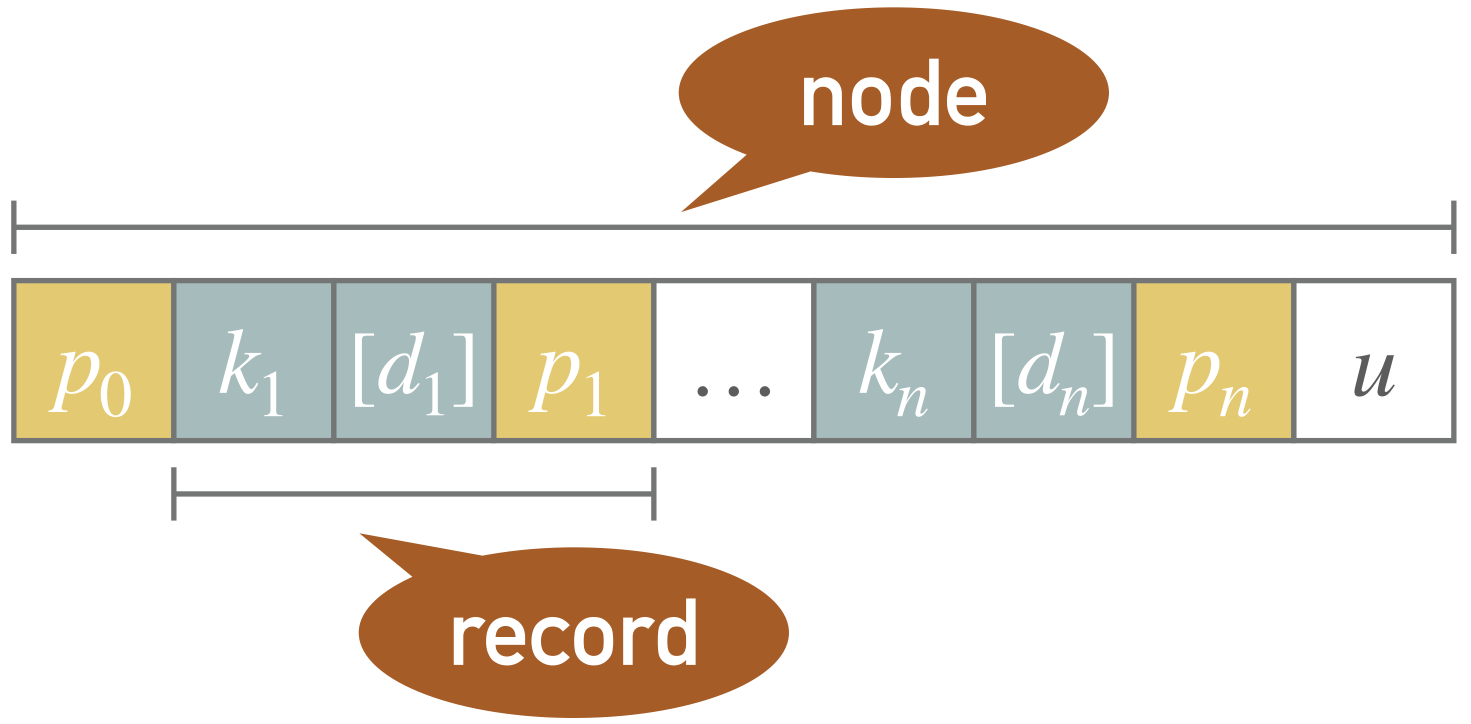
B-Trees

NDBI007: Practical class 5

B-Tree

- B-Tree of *degree* m is *balanced* m*-ary* tree where:
  - The *root* has at least 2 children unless it is a *leaf*
  - Every *inner node* have at least $\left\lceil \frac{m}{2} \right\rceil$ and at most m *children*
  - Every inner node contains at least $\left\lceil \frac{m}{2} \right\rceil - 1$ and at most $m - 1$ data entries (e.g., keys, pointers)
  - All the *paths* from the root to the leaf are of *the same length*

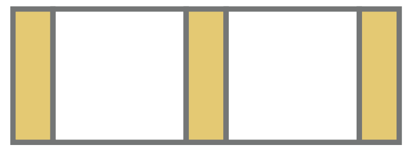
- The nodes have the structure $p_0, (k_1[, d_1], p_1), (k_2[, d_2], p_2), \ldots, (k_n[, d_n], p_n), u$
  - p_i - *pointers* to the children
  - k_i - *keys*
  - d_i - *data* or pointers to them
  - u - unused space
  - where $\left\lceil \frac{m}{2} \right\rceil - 1 \leq n \leq m - 1$



- Records $(k_i[, d_i], p_i)$ are *sorted* with respect to k_i
- Keys k_i in the subtree pointed by p_i are greater than or equal to k_i and less than i_{i+1}

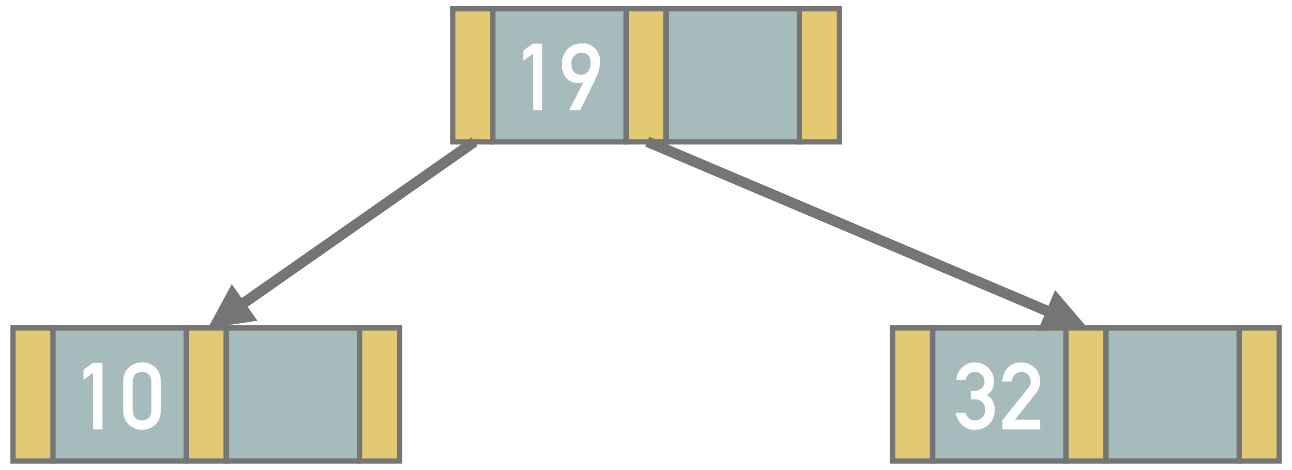
Example 5.1: Insert (Splitting the Root)

- Insert entries with keys 19, 10, and 32 into an empty tree
  - Suppose a non-redundant B-tree of degree $m = 3$
    - The inner nodes have between $\lceil 3/2 \rceil$ and 3 children, i.e., they contain between 1 and 2 keys



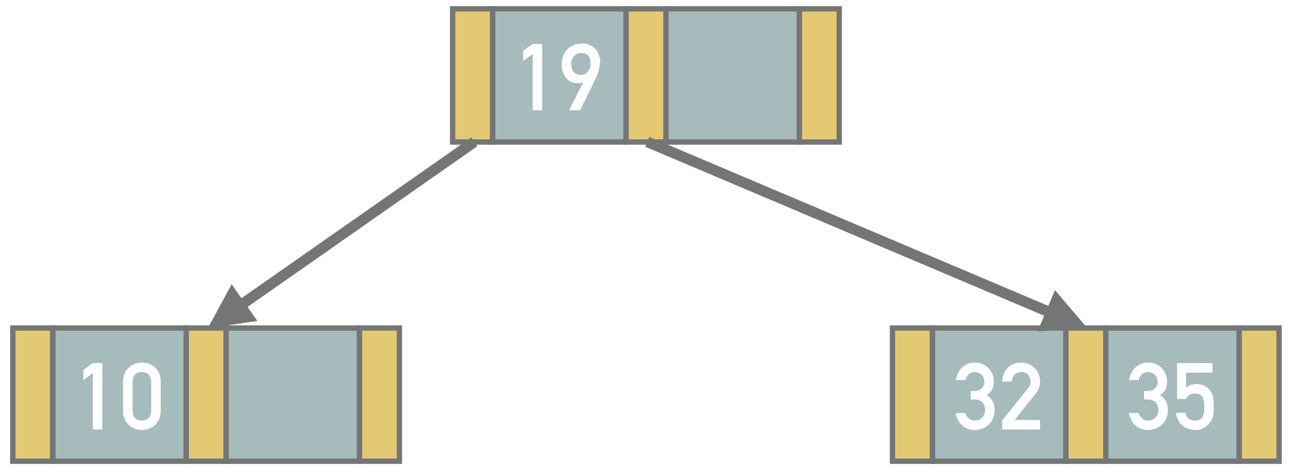
- The records with keys 19 and 10 fit into a single (root) node

- The record with key 32 does not fit and causes splitting
  - First, we order the keys 19, 10, and 32 in ascending order, i.e., 10, 19, and 32
  - The middle key (i.e., 19) will divide the smaller keys (i.e., 10) in one node from the bigger keys (i.e., 32) in a new node
  - The dividing key will be placed into the parent node (i.e., new root node)

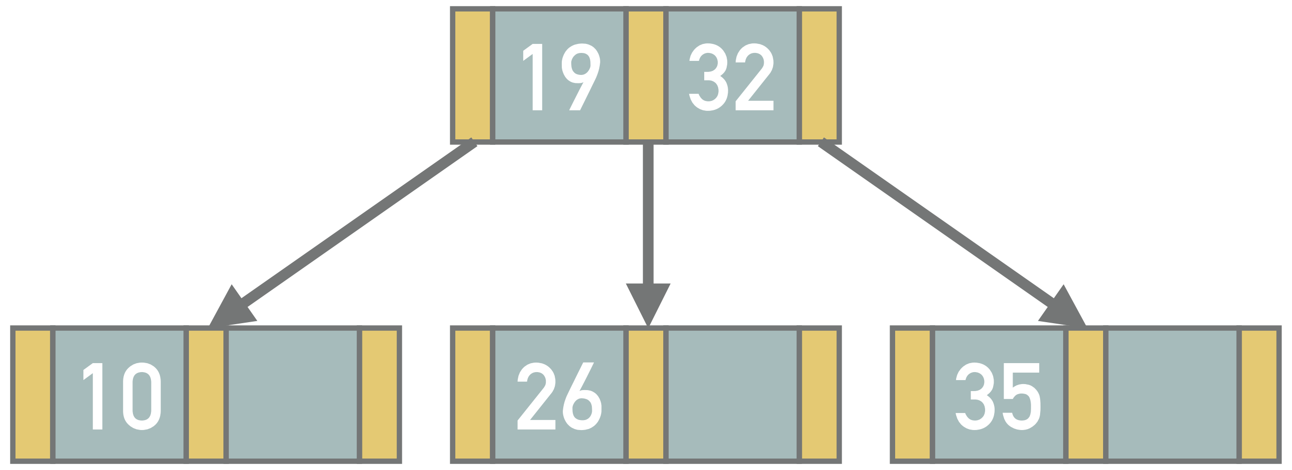


Example 5.2: Additional Inserts

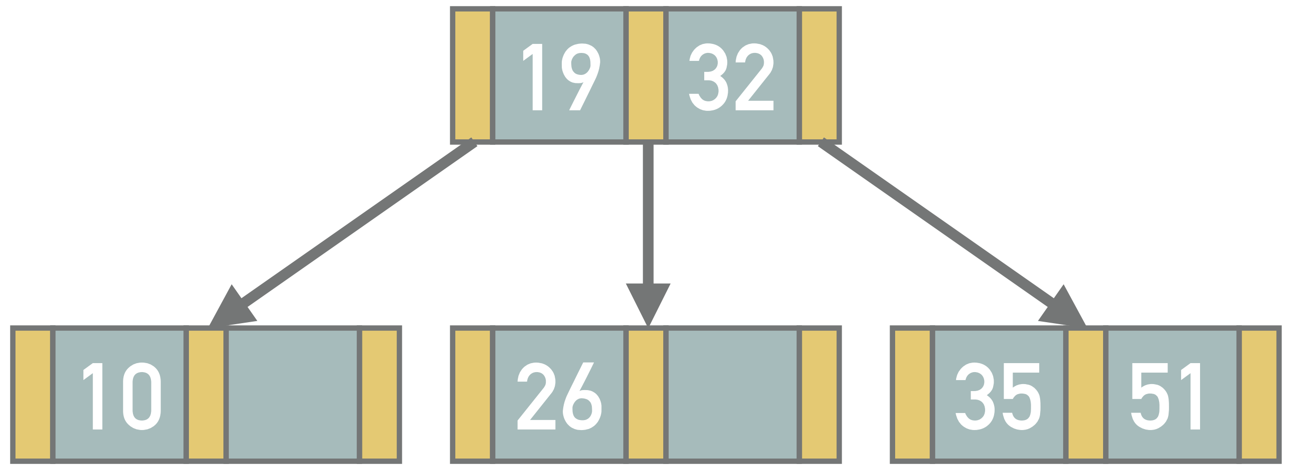
- Insert records with keys 35, 26, and 51 into B-tree from previous example



- The record with key 35 fits into the (right) leaf



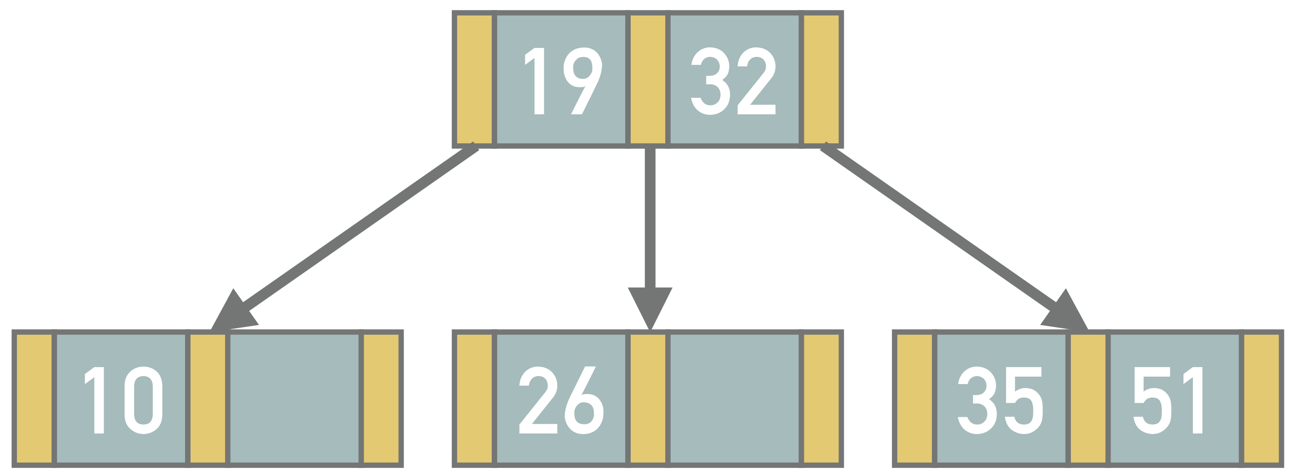
- The record with key 26 will split the (right) node into two nodes, i.e., (26) and (35) with (32) being the dividing record
    - The dividing record (23) finds its place in the parent node



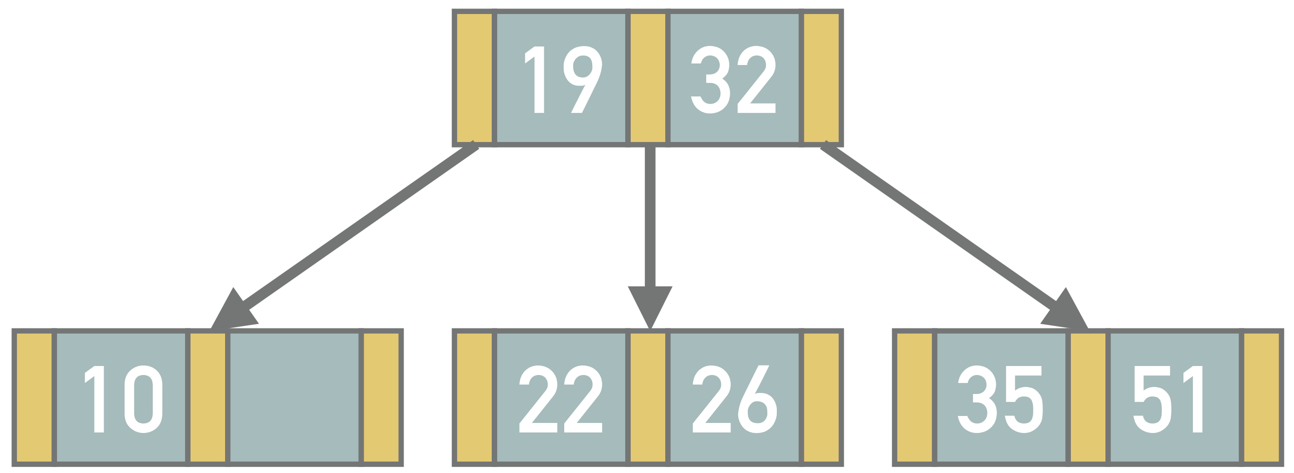
- The record 51 will fall into the right node

Example 5.3: Insert (Propagation)

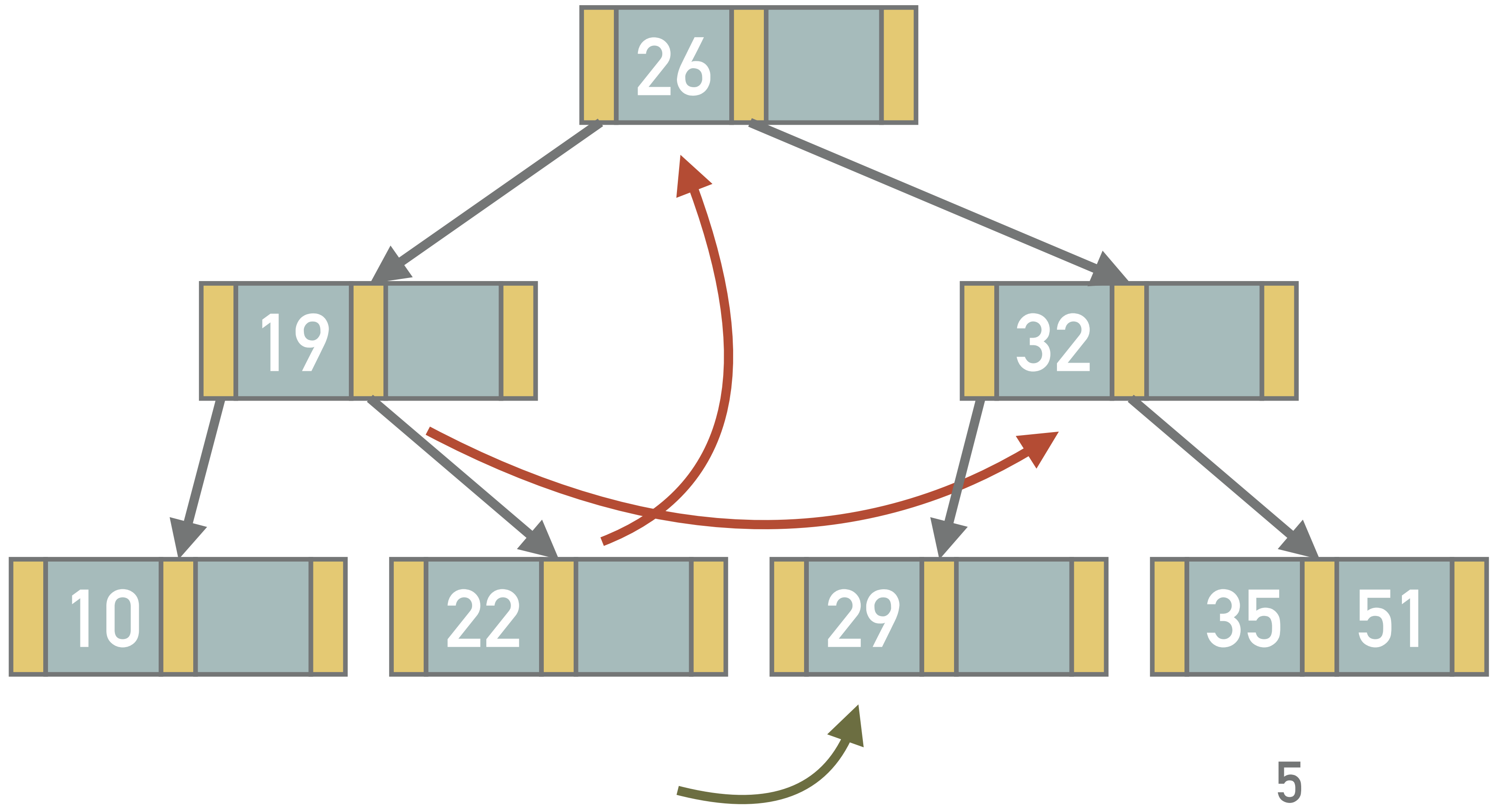
- Insert records with keys 22 and 29 into the B-tree from previous example



- The record 22 falls into the middle leaf

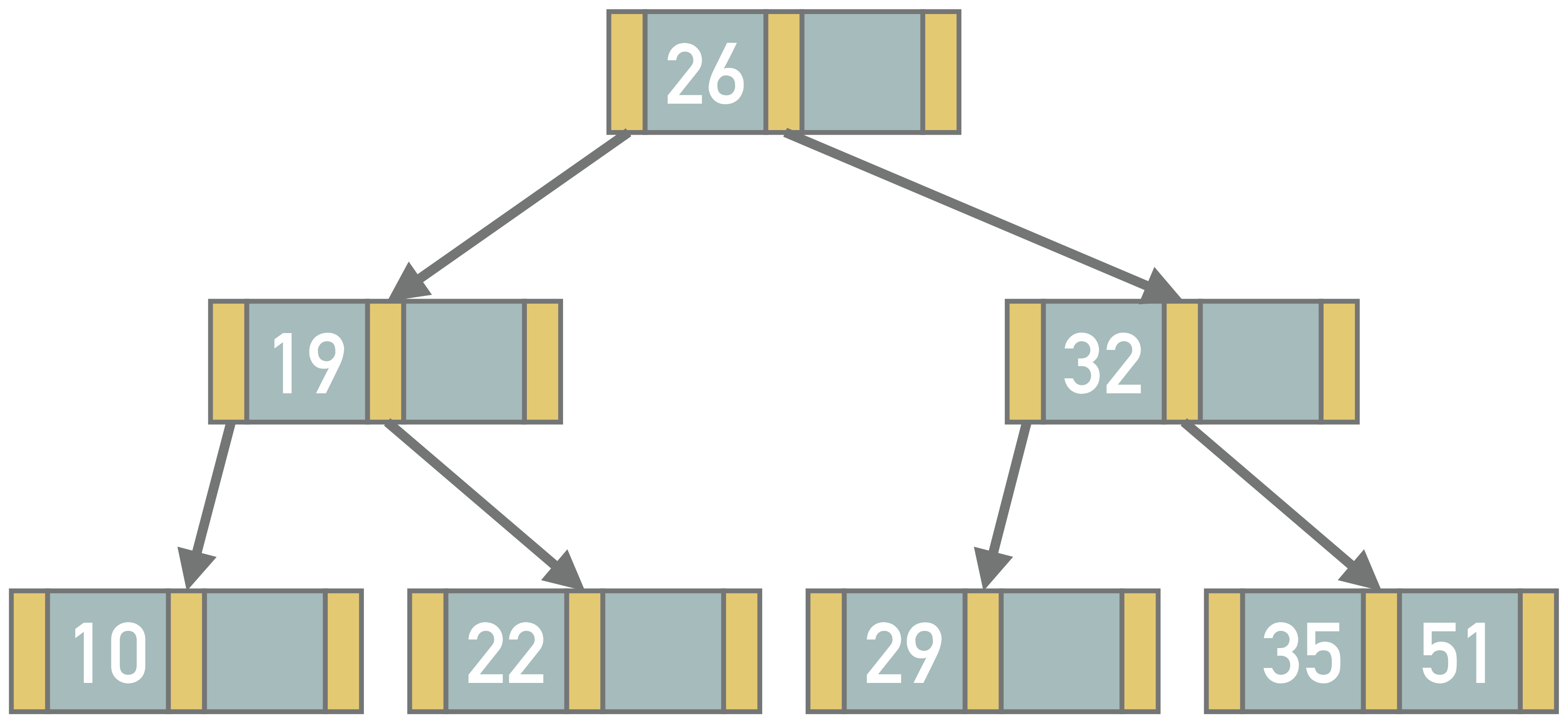


- The record 29 causes splitting of the middle leaf (22, 26, 29) and propagation of the record (26) to the parent
  - However, there is no more space in the parent node (root)
  - Thus, the parent node (19, 26, 32) needs to be split as well which increases the tree height

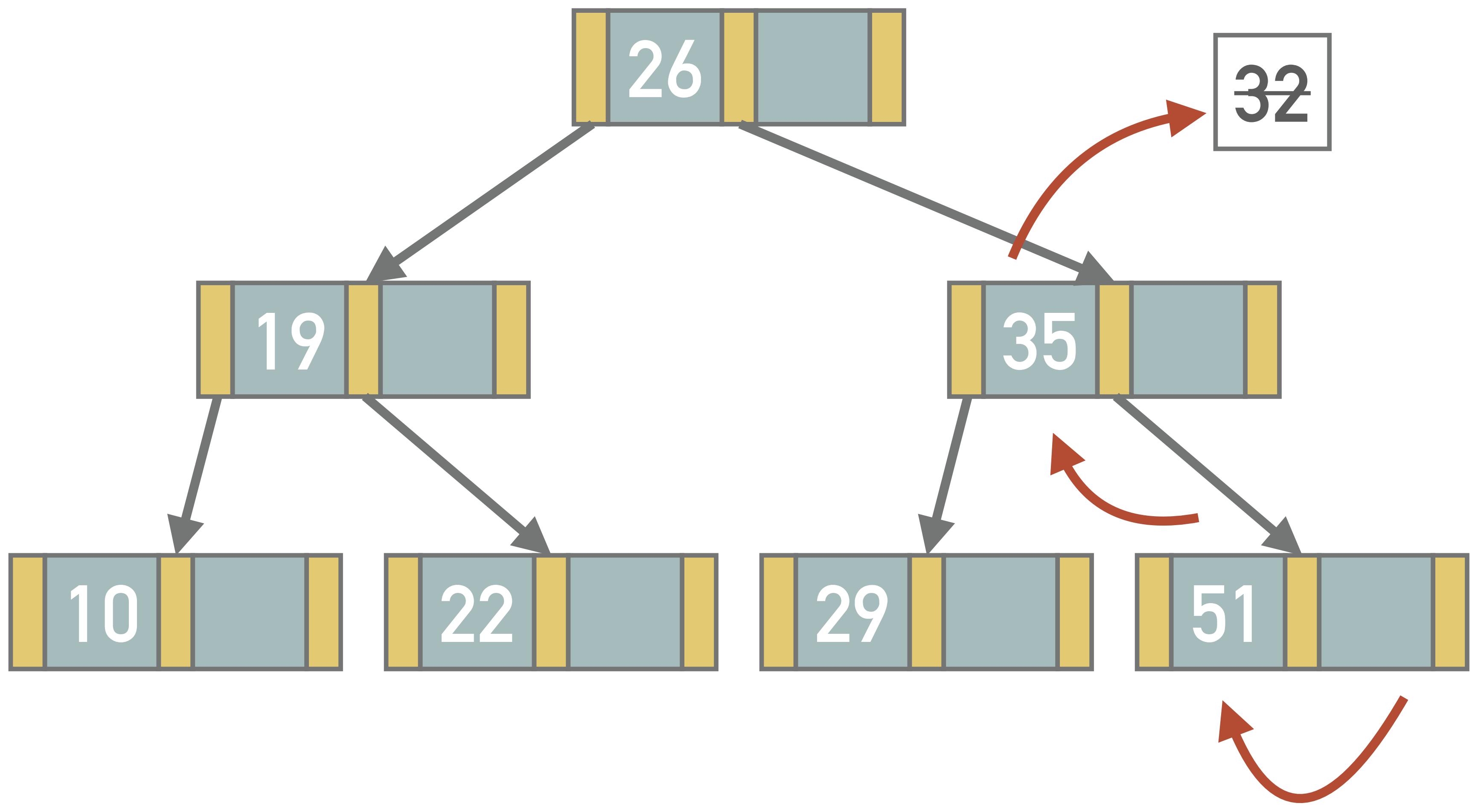


Example 5.4 Delete

- Remove record with key 32 from the non-redundant B-tree of degree 3 (see the upper figure)

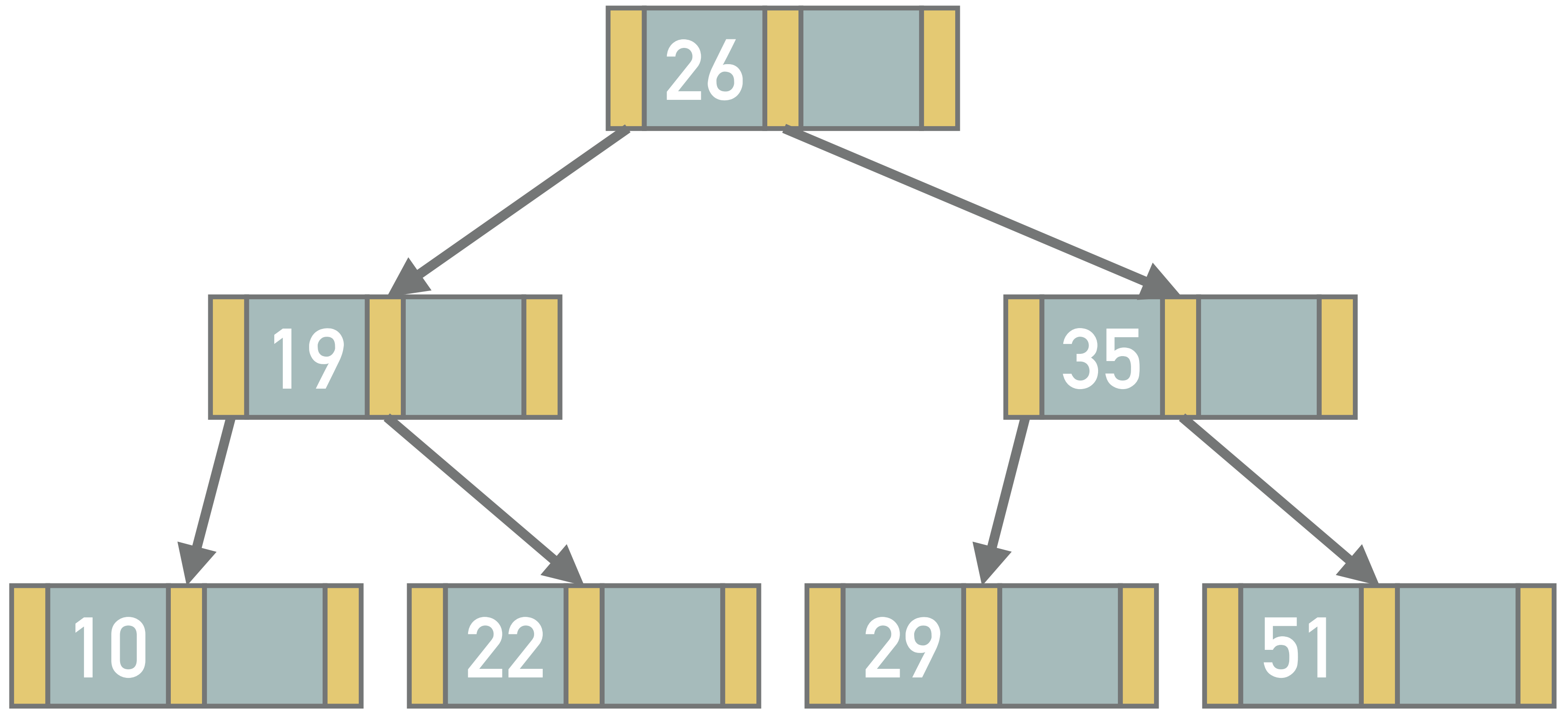


- The deletion of a data entry from an inner node leads to its replacement with the most left descendant entry from the right subtree or the most right entry from its left subtree
  - If we delete 32 from the tree above, we can replace it with entry 35 from the bottom node (leaf)
  - Moving the entry 35 from the leaf (35, 51) is safe since it still has the minimum number of entries

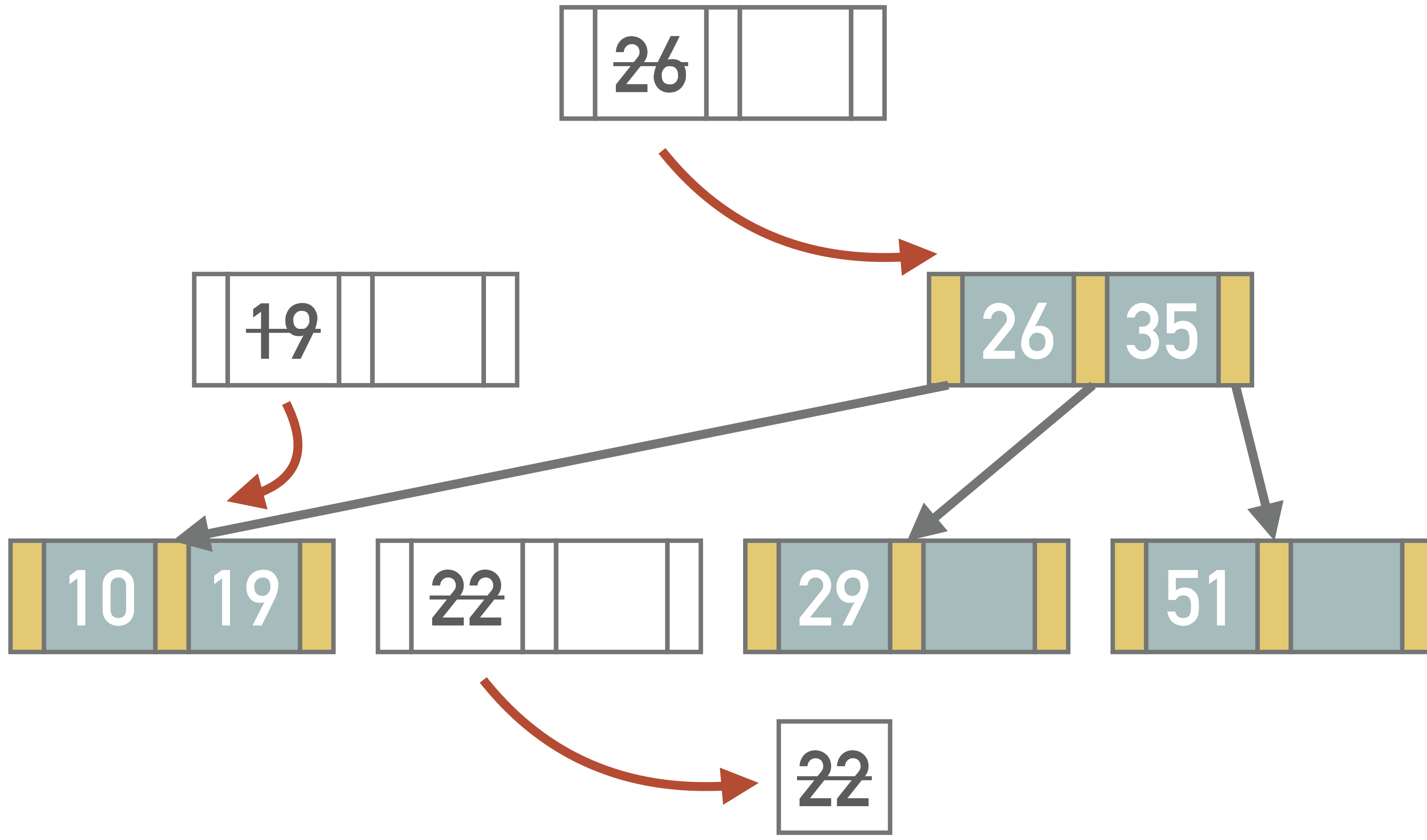


Example 5.5: Delete (Merging)

- Remove record with key 22 from the non-redundant B-tree of degree 3 (see the upper figure)



- We cannot borrow an entry from the neighbor (10) since it also contains the minimal number of entries
- Therefore we have to merge nodes (10), (empty), and (19)
  - The entries of the current node (none left after removing 22), those from the neighboring node (10) and the dividing node will be moved into a single node (10, 19)
    - Thus, the entry 19 needs to be removed from the parent node which causes underflow of that node
- We have to merge nodes (empty parent node), (26) and (35)
  - Once again, we cannot borrow an entry from the neighbor node (35)
  - The empty node (empty) is merged with the node (35) and dividing entry (26) from the root node, resulting in the node (26, 35)
  - Having entry 26 removed from the root (empty), the height of the tree decreases

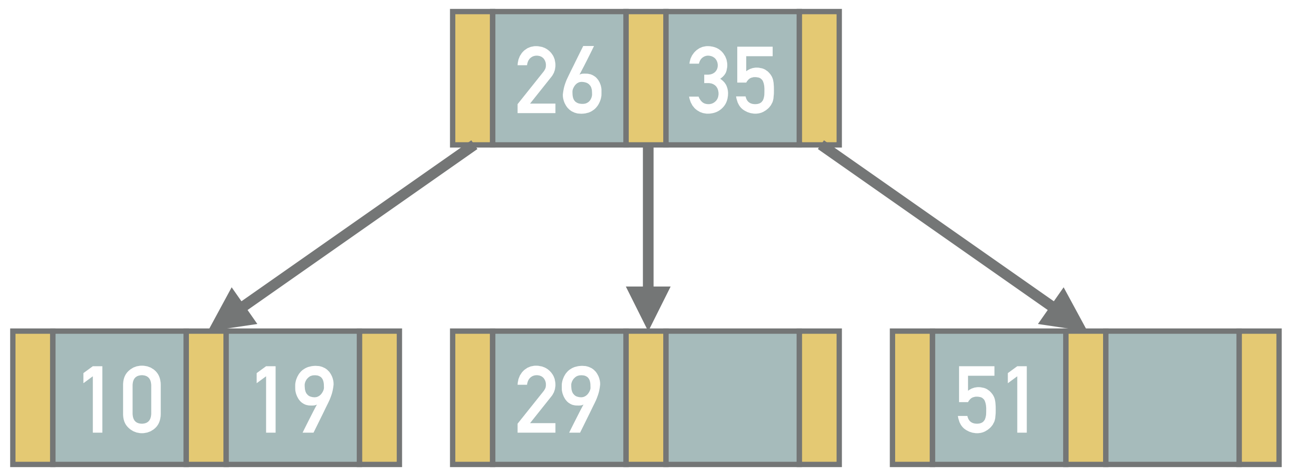


Exercise 5.6

- Suppose a non-redundant B-tree of degree $m = 3$ (see the figure)

- First, illustrate the B-tree after insertion of records with keys 13, 24, and 17

- Second, illustrate the B-tree after deletion of records with keys 51, and 17

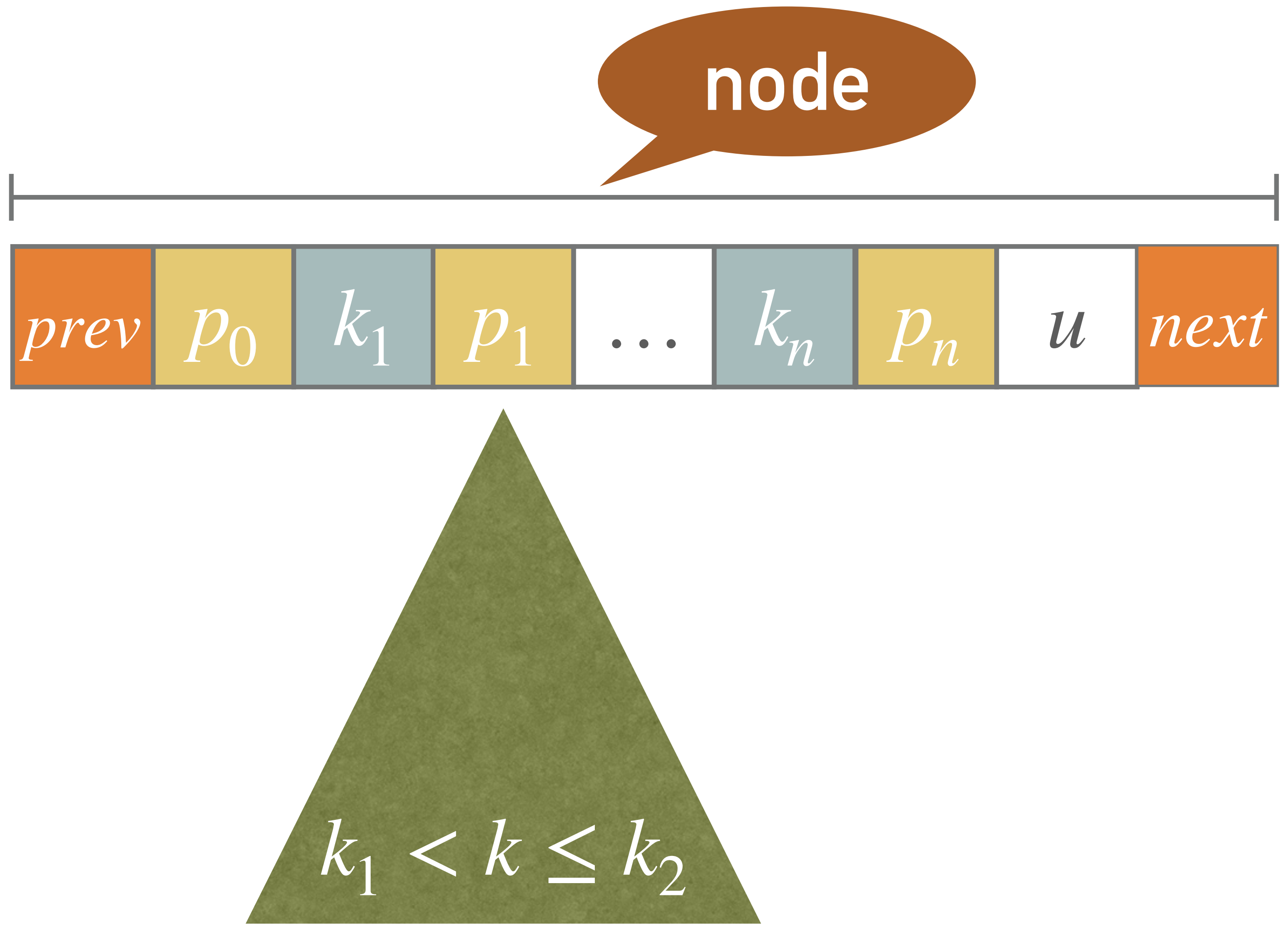


B+-Tree

- B+-Tree differs from the original B-tree by:
  - It is *always redundant*, i.e., the data are stored or pointed to from the leaf nodes
  - The *leaf nodes are chained* using pointers in a linked list which simplifies range queries
    - In reality, often all the levels are linked (not just the leaf level)
  - The inner nodes contain only the values using which the tree can be traversed

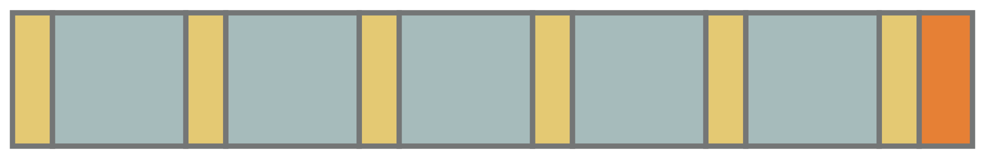
- The nodes have the structure $[prev,]\ p_0, (k_1, p_1), \ldots, (k_n, p_n), u\ [,next]$
- p_i - pointers to the children or data
- k_i - keys

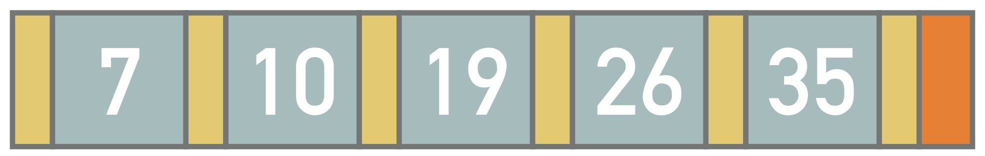
- Keys in the subtree pointed by p_i are greater than k_i and less than or equal to k_{i+1}, if k_{i+1} exists
- The minimum number of children can be raised to $\lceil (m+1)/2 \rceil$



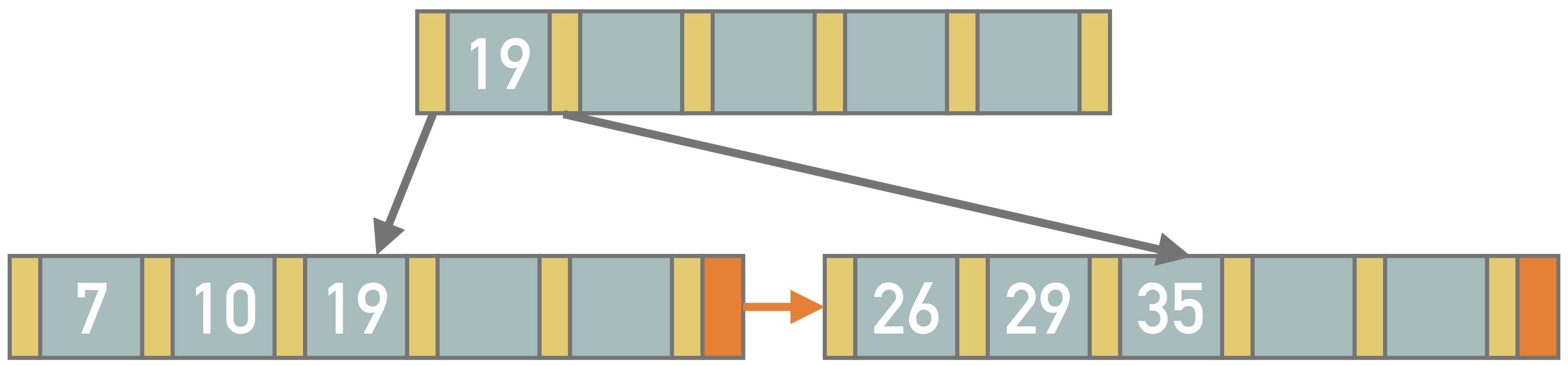
Example 5.7: Insert

- Insert records with keys 19, 10, 26, 7, 35, and 29 into an empty B+-tree
  - Suppose a B+-tree of degree $m = 6$
    - Hence, the minimum number of children is 3+1 (modified)





- Insertion of keys 19, 10, 26, 7, and 35 is trivial, all belong to the root node



- Insertion of key 29 leads to a page split
  - A half of the records, i.e., (7, 10, 19), stays in the original page while the rest, i.e., (26, 29, 35), moves into a new page
  - The maximal key value in the left node, i.e., 19, is propagated into the higher level (new root note)
    - However, any value $19 \leq value \leq 25$ would work

Example 5.8: Additional Inserts

- Insert additional records with keys 24, 1, 22, 32, 13, and 17 into the B+-Tree from the previous example

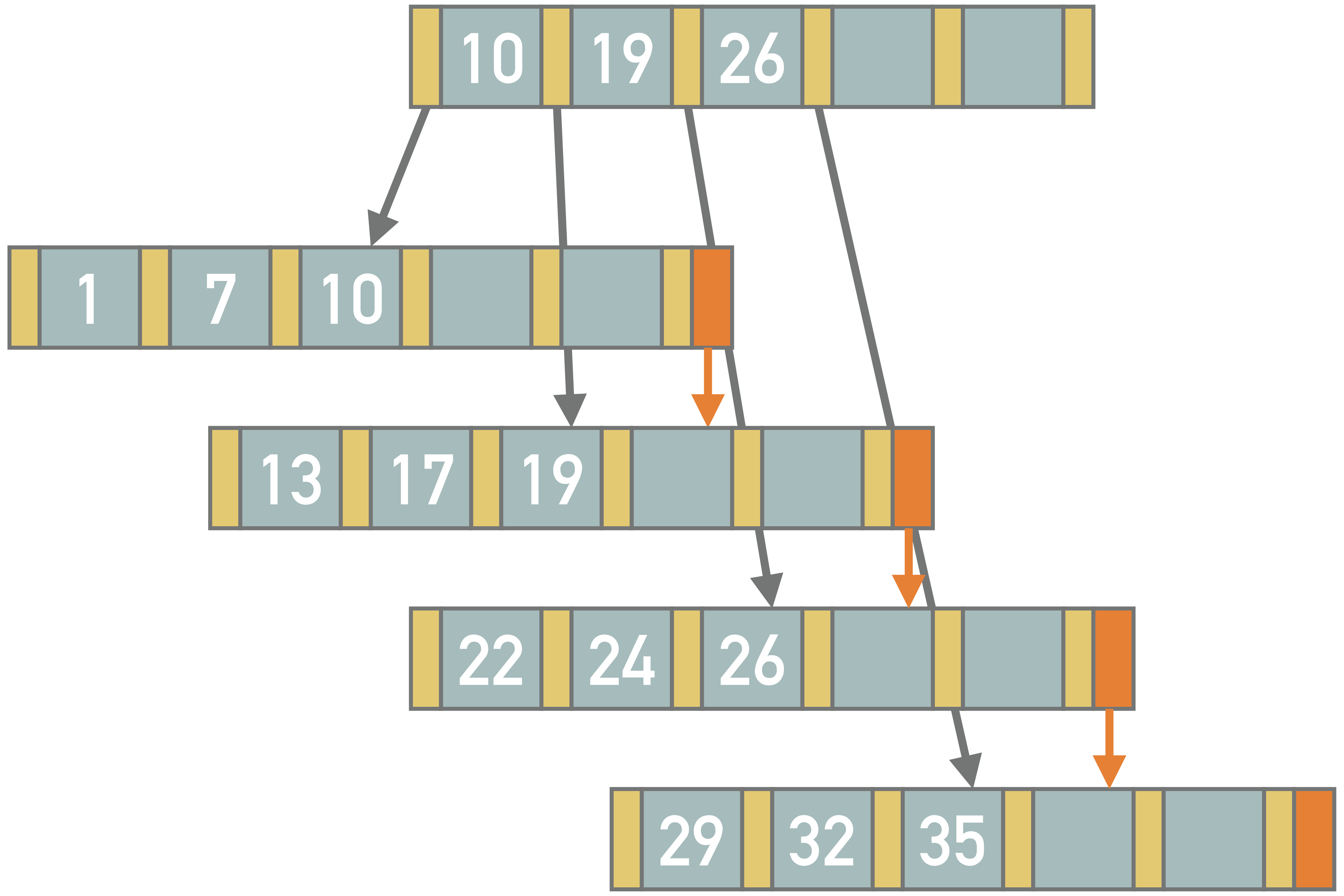
- The insertion of records with keys 24, 1, and 22 is trivial
- The insertion of a record with key 32 splits the right leaf node into nodes (22, 24, 26) and (29, 32, 35)
  - The separating value (i.e., 26) is inserted into the parent node where there is enough space so it does not lead to another split
- Inserting of records with keys 13 and 17 leads to the split of the leaf into (1, 7, 10) and (13, 17, 19)
  - The separation value 10 is inserted into the parent node

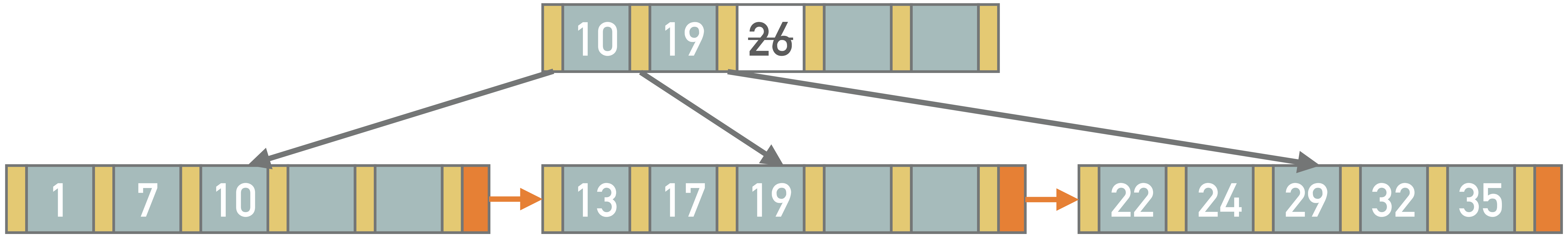
Example 5.9: Delete (and Merge Nodes)

- Remove the record with key 26 from the B+-Tree

- When removing keys from a B+-Tree, the given key is simply removed from the leaf unless the corresponding leaf underflows
  - In such case, the tree tries to borrow a key from a sibling leaf and to change the splitting value
  - If also the neighbors have the minimum number of entries, it is necessary to merge two nodes into one and remove the splitting value from the parent
    - Which can lead to the merge cascade up to the root

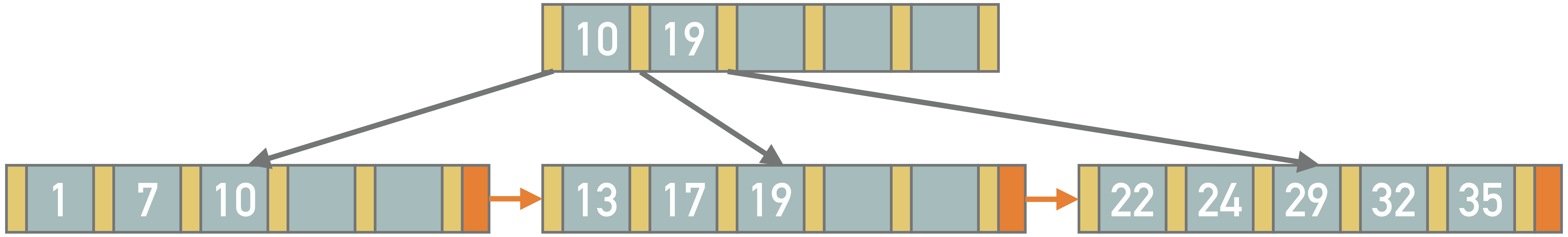
- In our example, every node (except the root) needs to include at least three keys
  - Removing the key 26, the condition is violated and sibling leaves cannot lose any entry either
  - Hence we merge node (22, 24) with (29, 32, 35) and remove the splitting value 26 from the parent



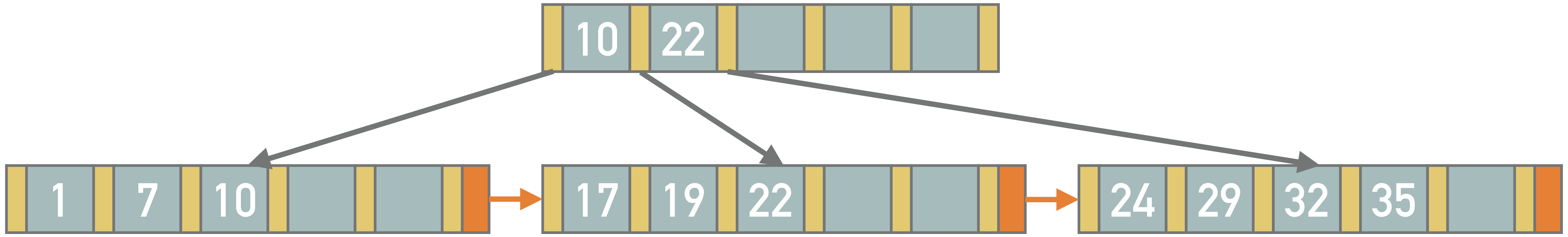


Example 5.10: Delete (Borrow Key)

- Remove the entry with key 13 from the B+-tree

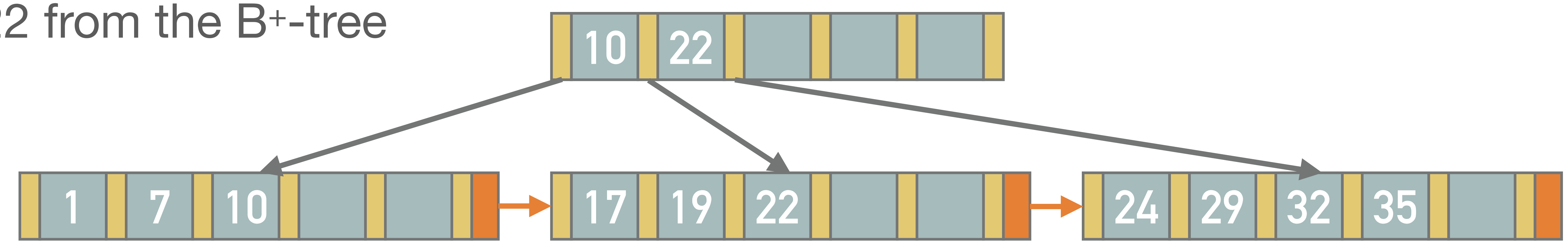


- To remove the entry 13 we need to move the entry with key 22 from the neighboring node to keep the condition of minimum number of entries in every node
  - It is necessary to change the splitting value in the parent from 19 to 22

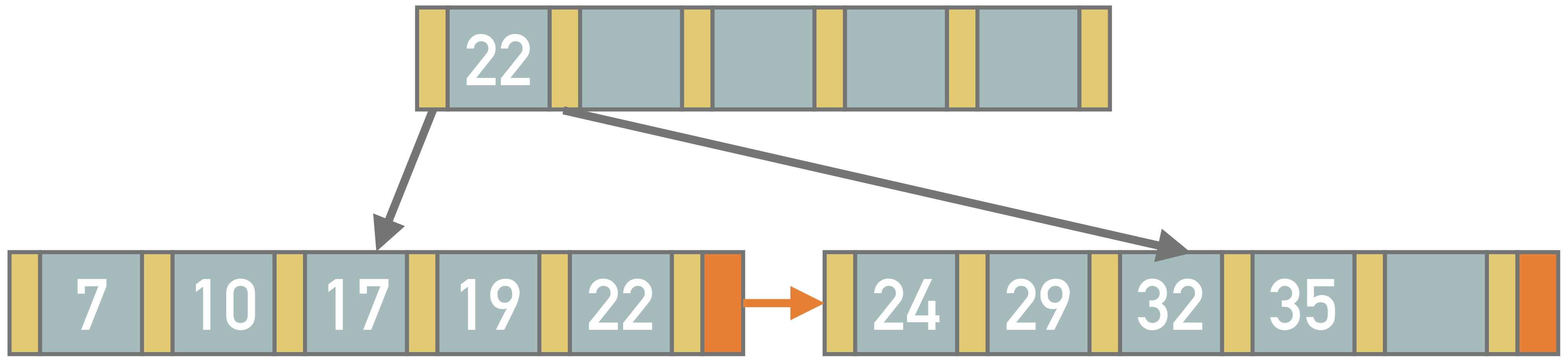


Example 5.11: Delete

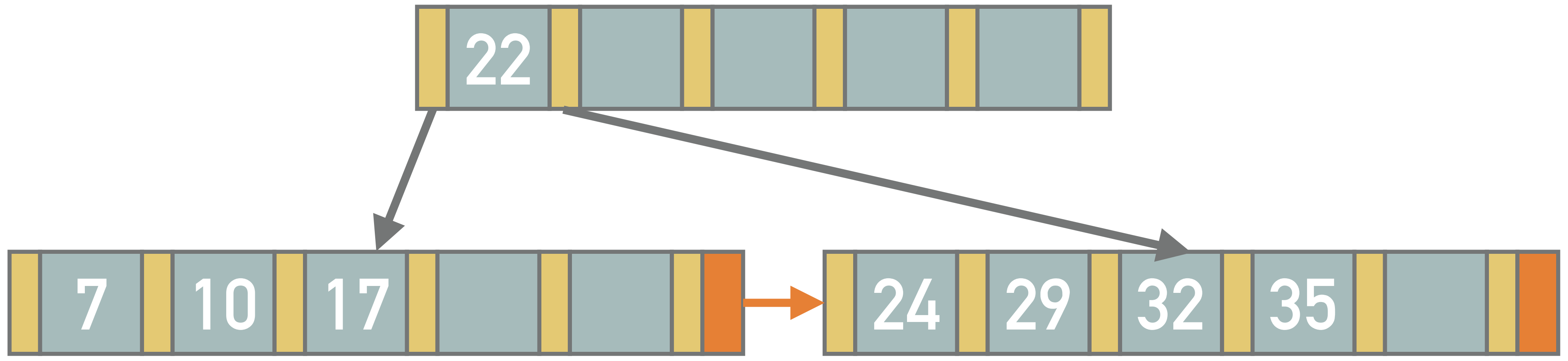
- Remove records with keys 1, 19, and 22 from the B+-tree



- Removing the key 1
  - After the removal, the number of records in the node (7, 10) falls under minimum and the neighboring node, i.e., (17, 19, 22), cannot provide any record
  - The nodes (7, 10) and (17, 19, 22) are merged
  - Finally, the splitting value 10 is removed from the parent



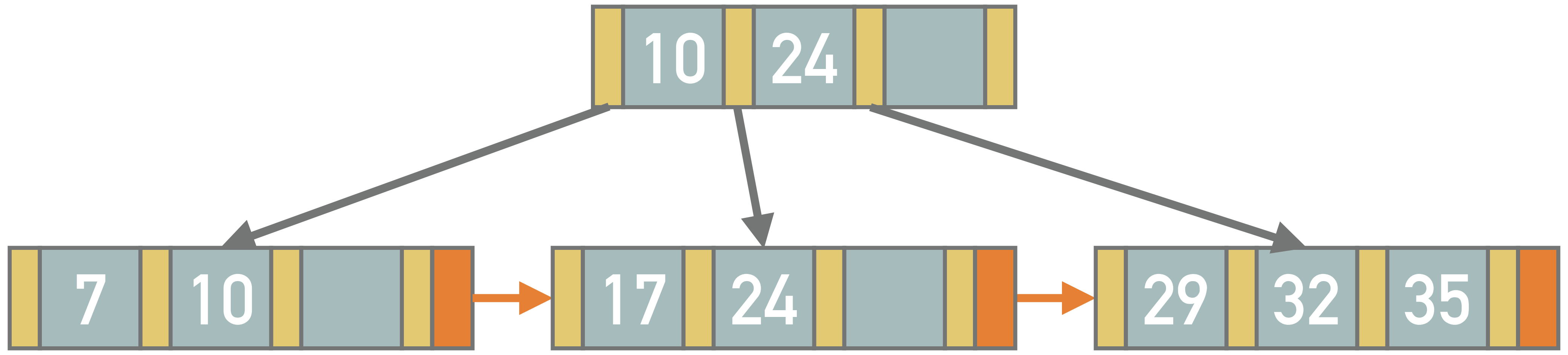
- Removing the keys 19, 22
  - It is sufficient to remove the keys from the node, no modifying of splitting value is required



Exercise 5.12

- Suppose a B+-tree of degree $m = 4$ (see the figure)
  - Minimum modified number of children of a node is 3, i.e., $\lceil (4+1)/2 \rceil$

- Illustrate the B+-tree after the insertion of keys 51, 80, and 99

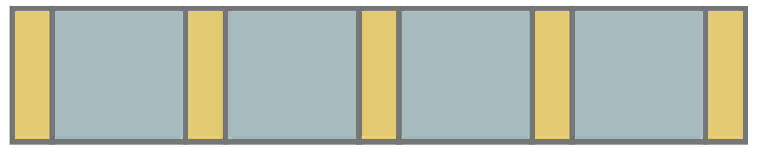


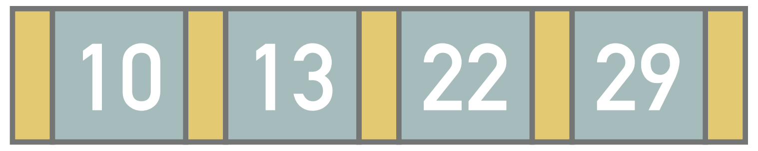
B*-Tree

- B*-tree differ from the standard B-tree by:
  - The non-root nodes have at least $\lceil (2m-1)/3 \rceil$ children
    - If the tree contains few records (i.e., after splitting the root node), the only two leafs can contain less records (about half)
  - If a node has too few items, or overflows, it is balanced using both of its neighbors
  - If a node and its neighbor are full, they are split (together with the new record) into three nodes being 2/3 filled

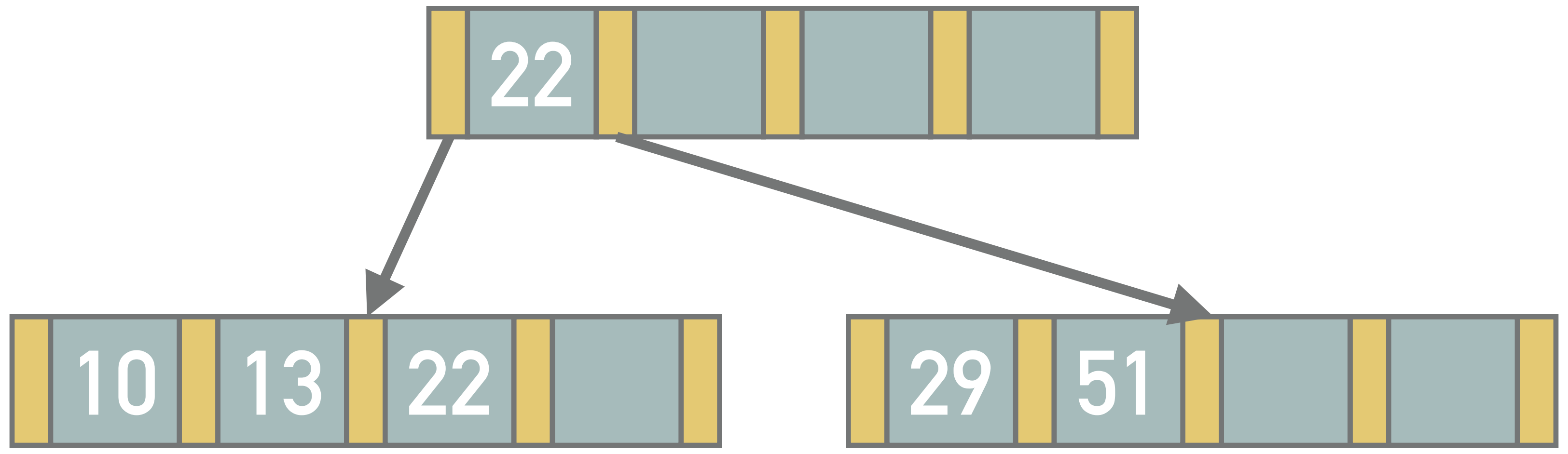
Example 5.13: Insert

- Insert records with keys 22, 13, 29, 10, 51, 32, and 26 into an empty redundant B*-tree
  - Suppose an empty B*-tree of degree $m = 5$
  - Minimum number of children is 3 and minimum number of keys is 2

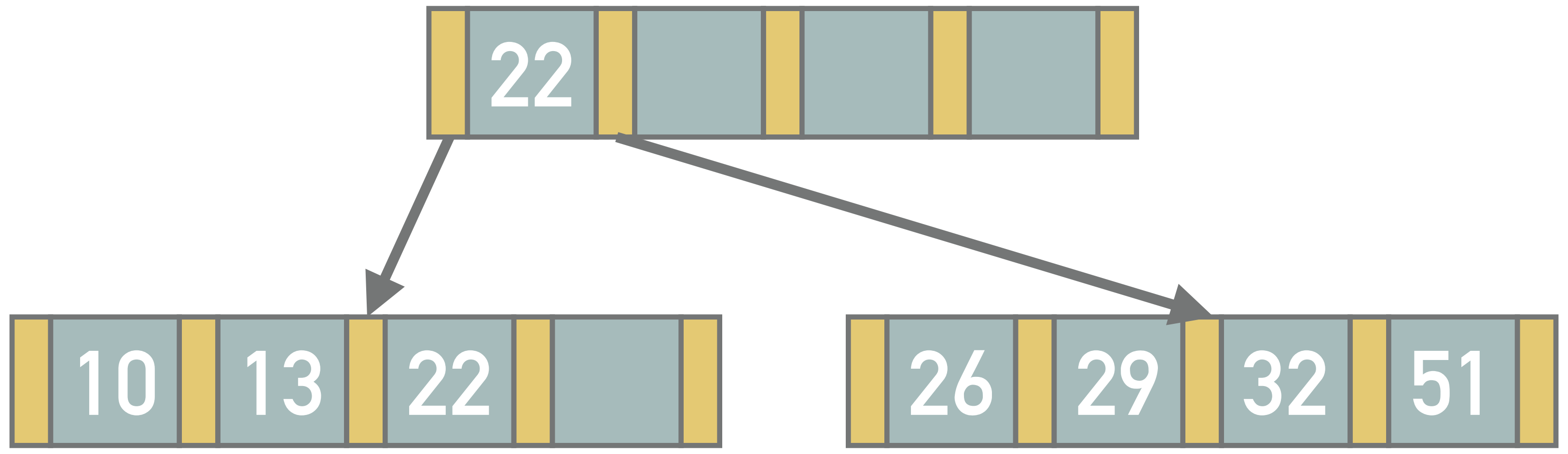




- Insertion of records with keys 22, 13, 29, and 10 is trivial, all goes to the root node
- Inserting a record with key 51 leads to root node split
  - Split nodes are (10, 13, 22) and (29, 51)
  - The dividing value 22 is inserted into the new parent (i.e., new root)



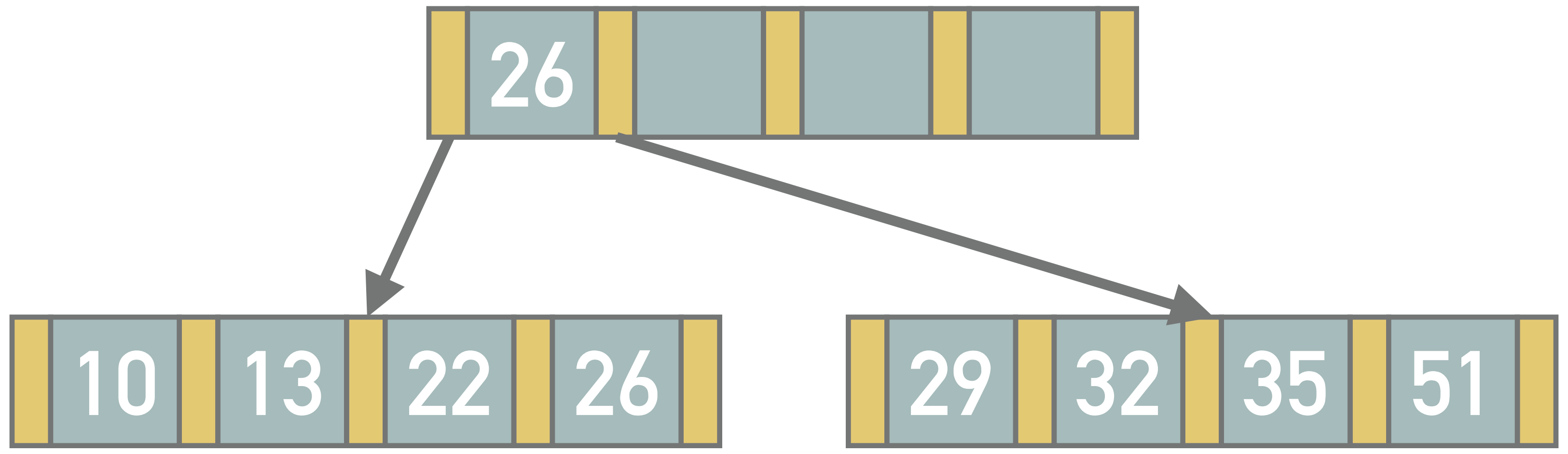
- A record with key 32 can be inserted into the right leaf, as well as a record with a key 26



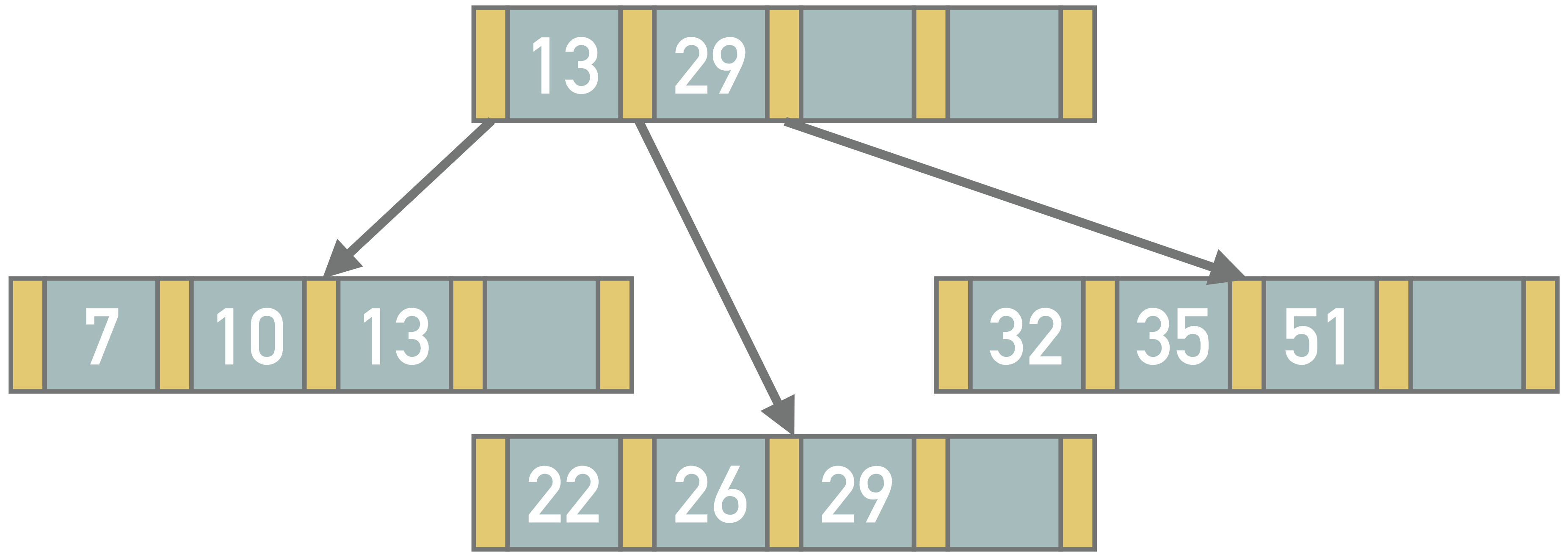
Example 5.14: Additional Inserts

- Continue with the previous example and insert records with keys 35 and 7 into the redundant B*-tree

- Inserting the key 35
  - We cannot insert the key 35 into the full node (26, 29, 32, 51), but the record with key 26 can be moved to the neighboring and not yet filled node
  - The splitting value in the parent needs to be modified

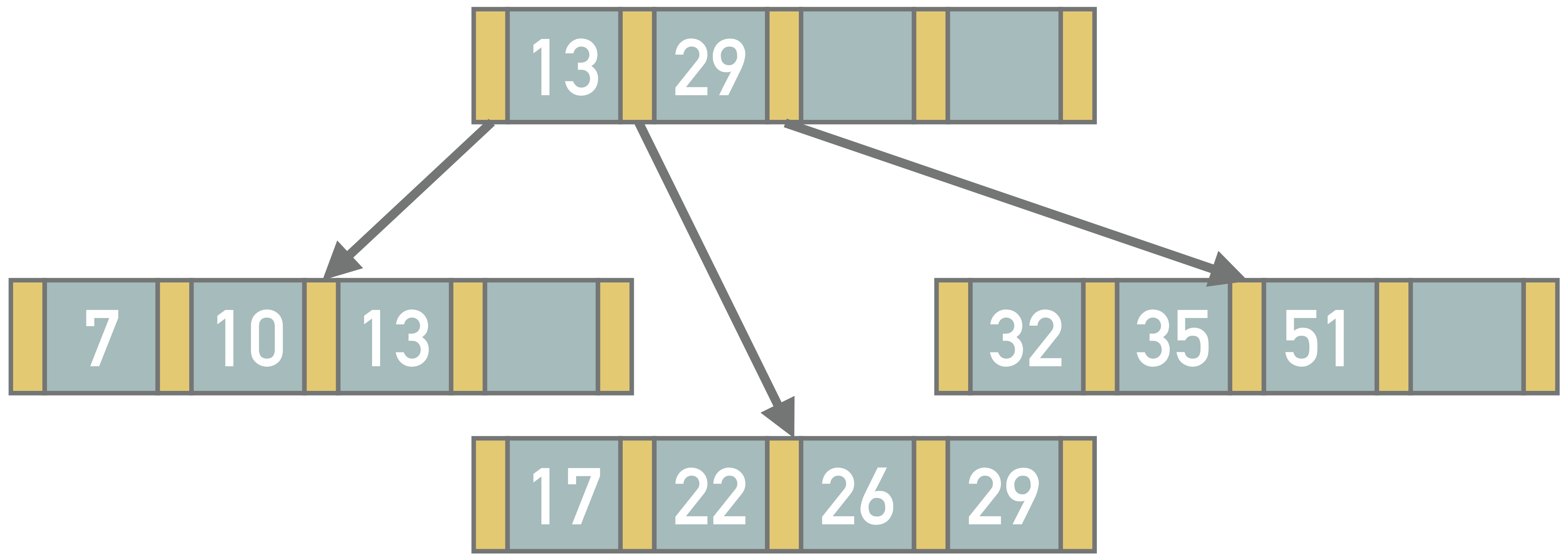


- Inserting the key 7
  - The key 7 cannot be inserted into the node (10, 13, 22, 26) and the neighbor is full as well
  - The records in both nodes, together with record 7, will be split into three nodes (7, 10, 13), (22, 26, 29) and (32, 35, 51)
  - Splitting values 13 and 29 need to be inserted into the parent node instead of the existing splitting value 26



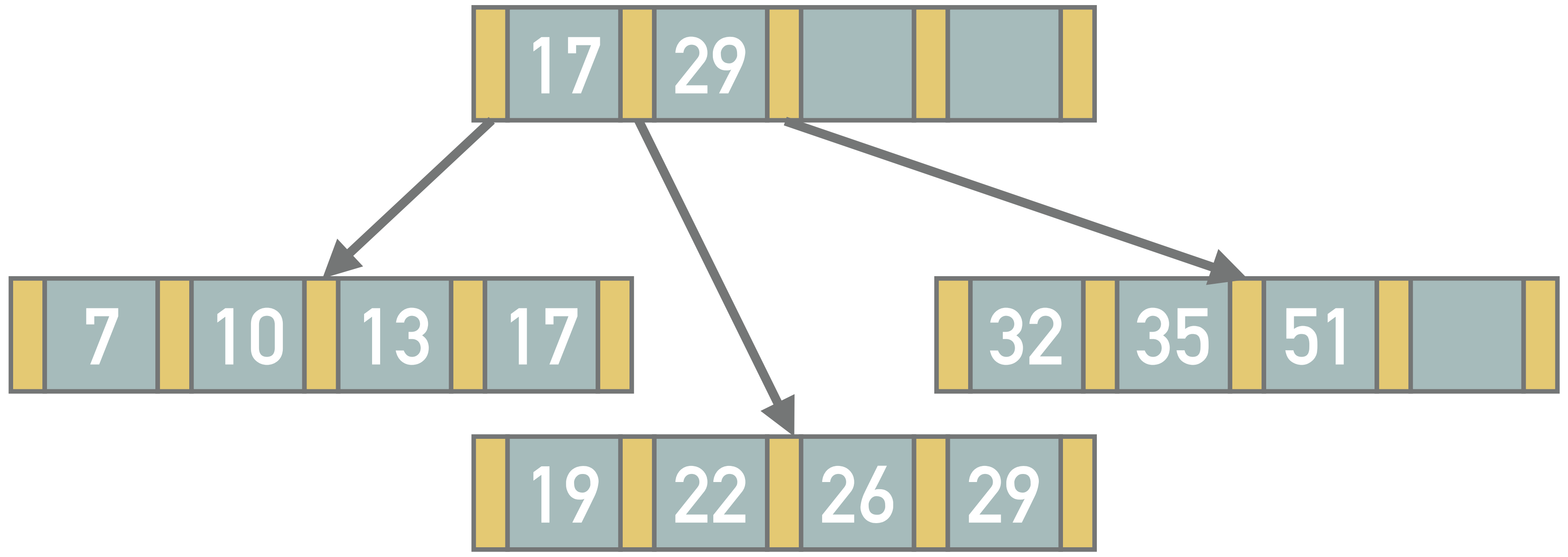
Example 5.15: Additional Inserts

- Continue with the previous example and insert records with keys 17, 19, and 24 into the redundant B*-tree

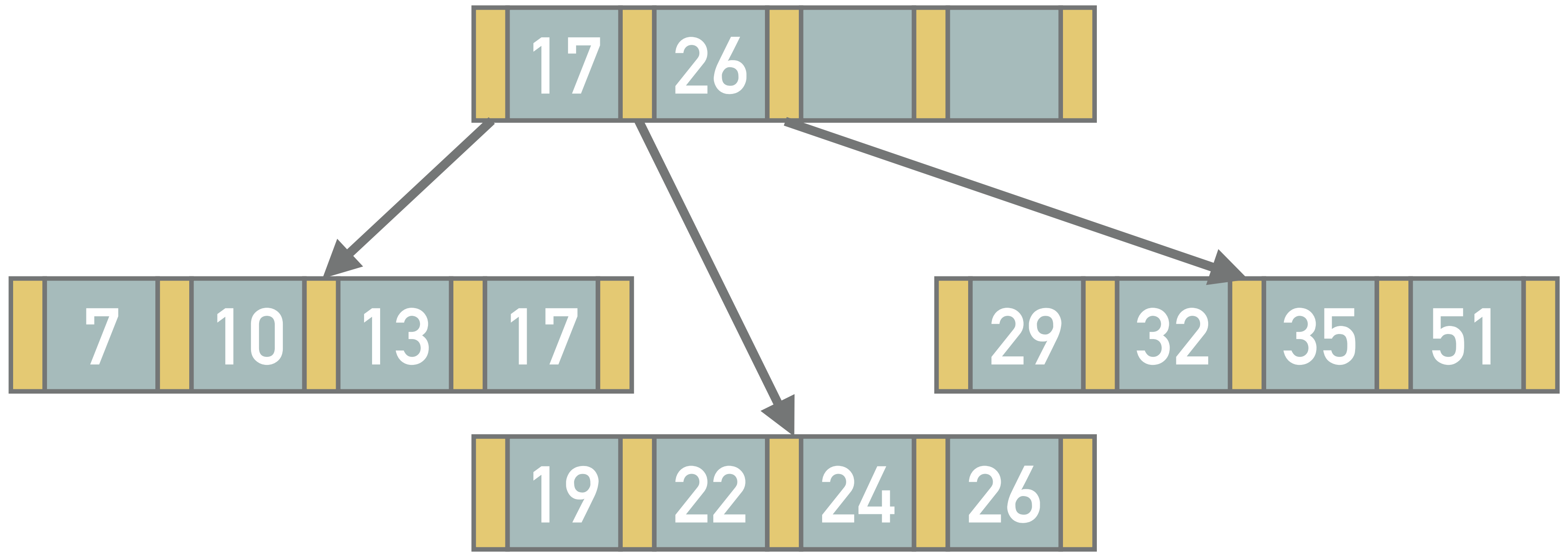


- The record 17 fits into the middle leaf

- The record 19 causes redistribution of the record 17 to the leaf and change of the splitting value from 13 to 17

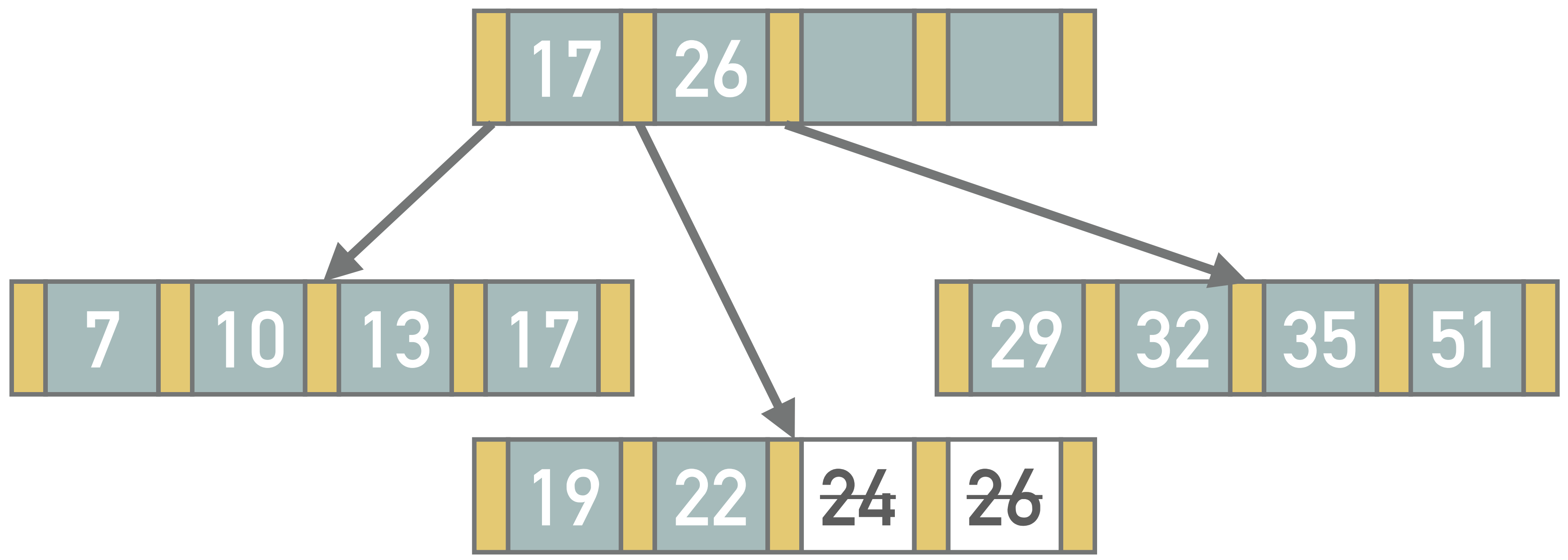


- The record with a key 24 will cause one of two possibilities:
  - **The redistribution of the record with key 29 to the right and modification of the splitting value in the parent from 29 to 26**
  - Split of nodes (7, 10, 13, 17) and (19, 22, 26, 29) into three nodes (7, 10, 13), (17, 19, 22) and (24, 26, 29)
    - The splitting value 17 would be replaced by a pair 13 and 22



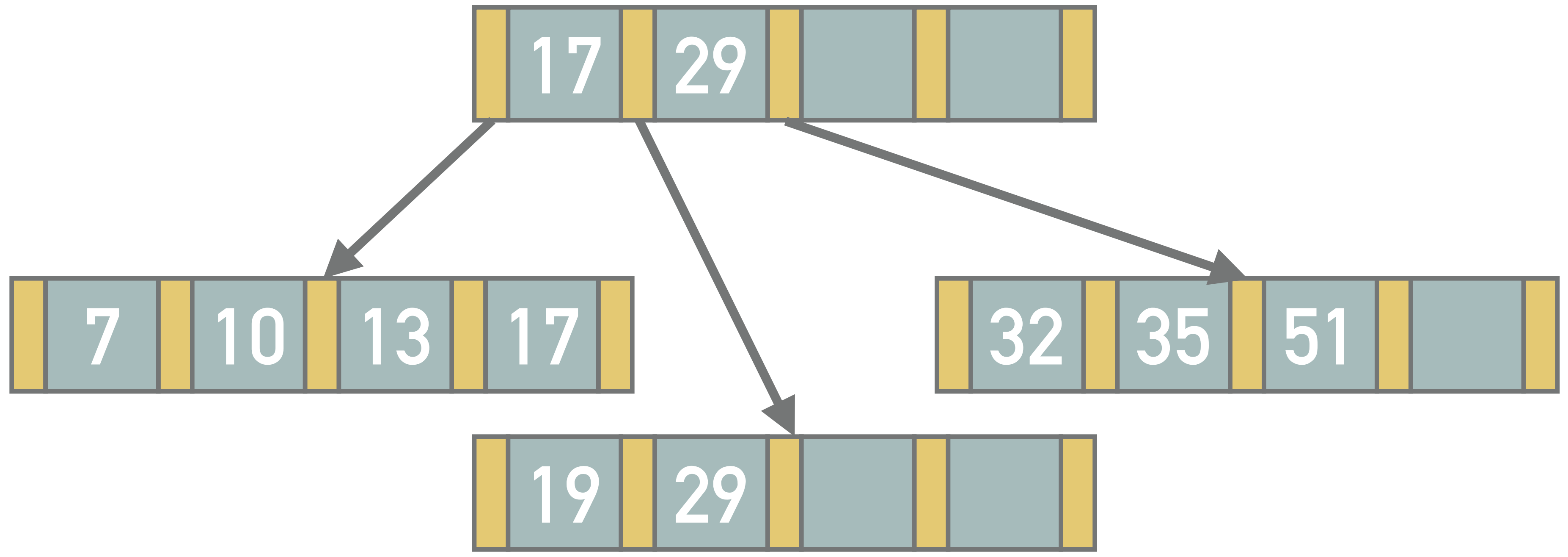
Example 5.16: Delete

- Continue with previous example and delete the records with keys 26, 24, and 22 from redundant B*-tree



- The record with key 26 can be easily deleted from the middle leaf
- The same holds for the record with key 24

- The record with key 22 cannot be deleted directly
  - The number of entries in a node would decrease under the threshold
  - Therefore it is necessary to move there the record with key 29 from the neighboring node
  - The splitting value in the parent changes from 26 to 29



Exercise 5.17

- Continue with previous example and delete records with keys 29, 19, and 17 from redundant B*-tree (see the figure)
- Finally, remove (single) additional key of your choice from the B*-tree
  - Illustrate and comment the removals step by step