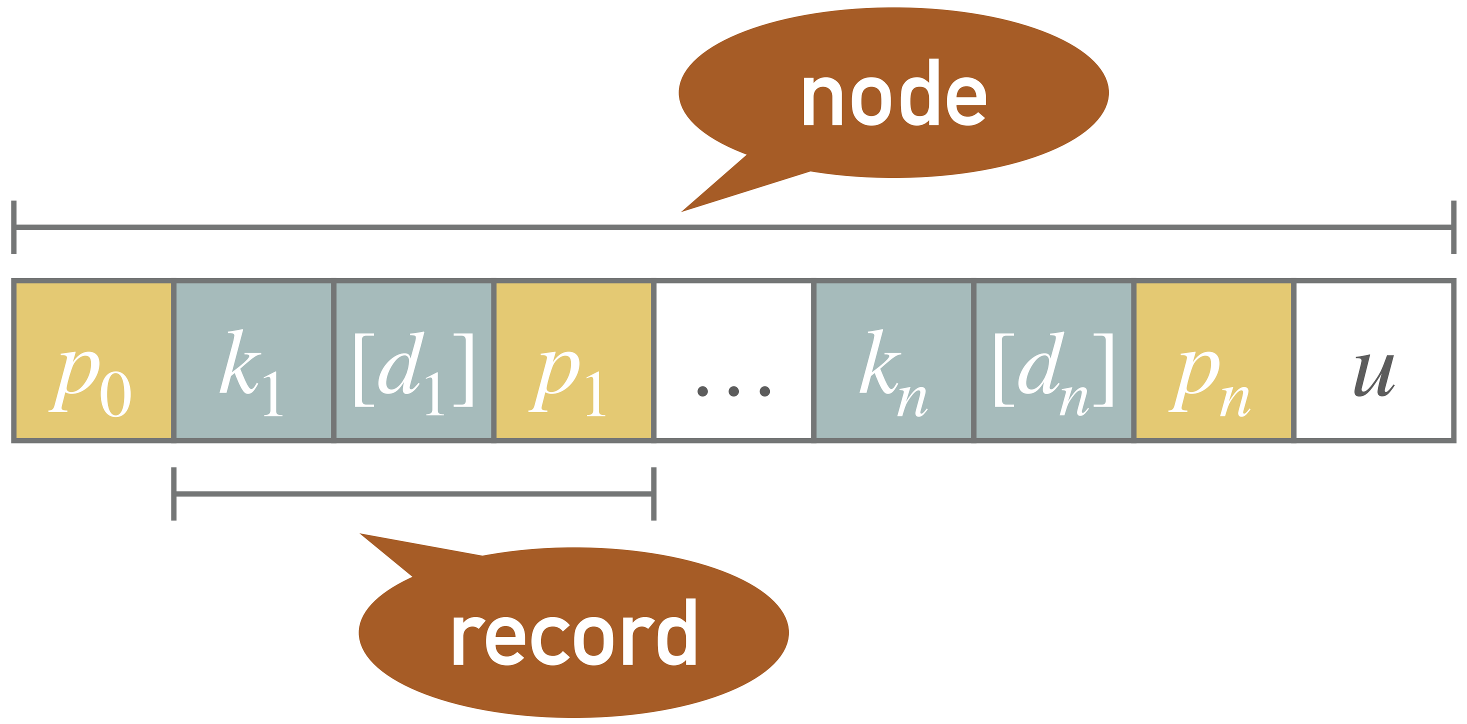
B-TREES

NDBI007: Practical Class 5

B-TREE

- B-Tree of *degree* m is *balanced m-ary* tree where
  - The *root* has at least 2 children unless it is a *leaf*
  - Every *inner nodes* have at least $\lceil \frac{m}{2} \rceil$ and at most m *children*
  - Every inner node contains at least $\lceil \frac{m}{2} \rceil - 1$ and at most $m - 1$ data entries (e.g., keys, pointers)
  - All the *paths* from the root to the leaf are of *the same length*

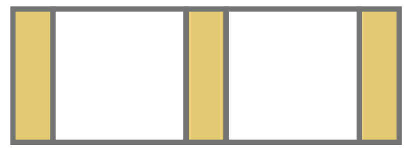
- The nodes have the structure $p_0, (k_1[, d_1], p_1), (k_2[, d_2], p_2), \ldots, (k_n[, d_n], p_n), u$
  - p_i - *pointers* to the children
  - k_i - *keys*
  - d_i - *data* or pointers to them
  - u - unused space
  - where $\lceil \frac{m}{2} \rceil - 1 \leq n \leq m - 1$

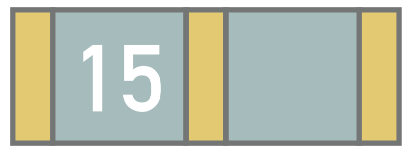


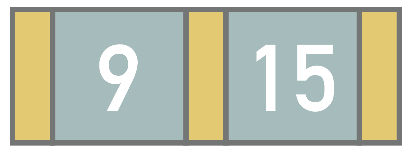
- Records $(k_i[, d_i], p_i)$ are *sorted* with respect to k_i
- Keys k_j in the subtree pointed by p_i are greater than or equal to k_i and less than k_{i+1}

EXAMPLE 1: INSERT (SPLITTING THE ROOT)

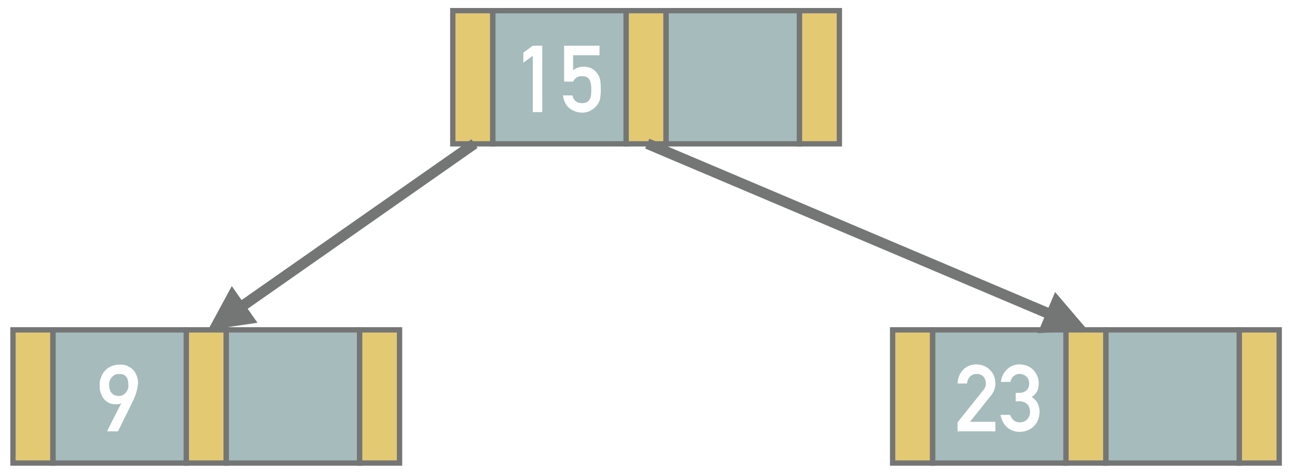
- Insert entries with keys 15, 9 and 23 into an empty tree
  - Suppose a non-redundant B-tree of degree $m = 3$
    - The inner nodes have between $\lceil 3/2 \rceil$ and 3 children, i.e., they contain between 1 and 2 keys







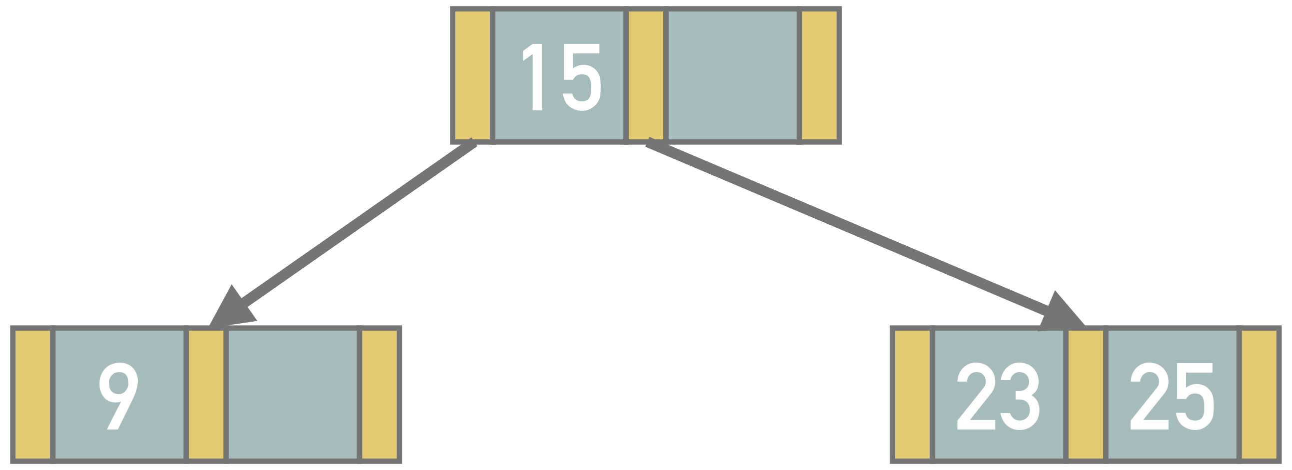
- The records with keys 15 and 9 fit into a single (root) node



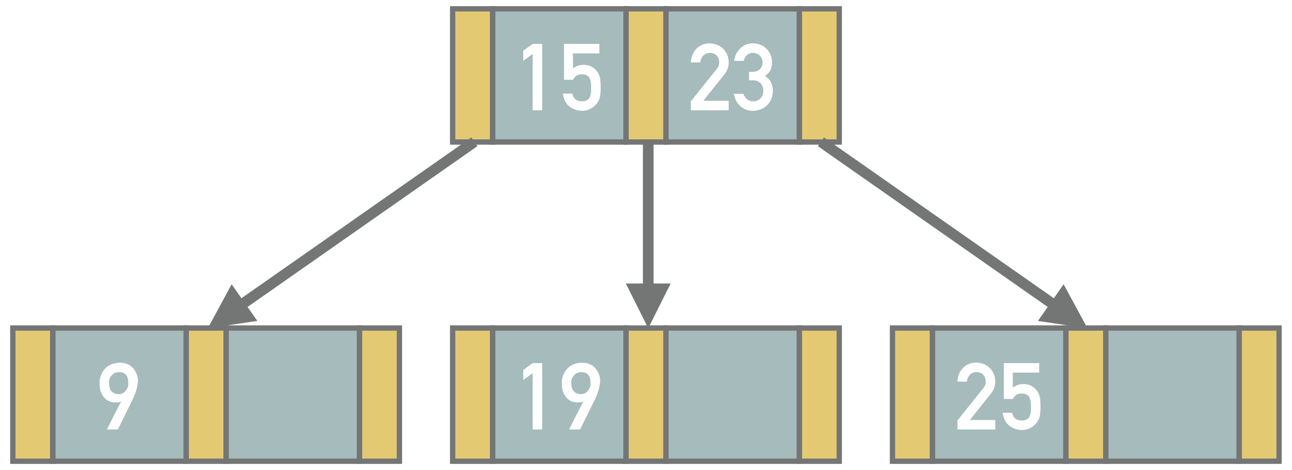
- The record with key value 23 does not fit and causes splitting
  - First, we order the records 15, 9, 23 to 9, 15, 23 (ascending order)
  - The middle record (15) will divide the smaller records (9) in one node from the bigger records (23) in a new node
  - The dividing record will be placed into the parent node (new root node)

EXAMPLE 2: INSERT

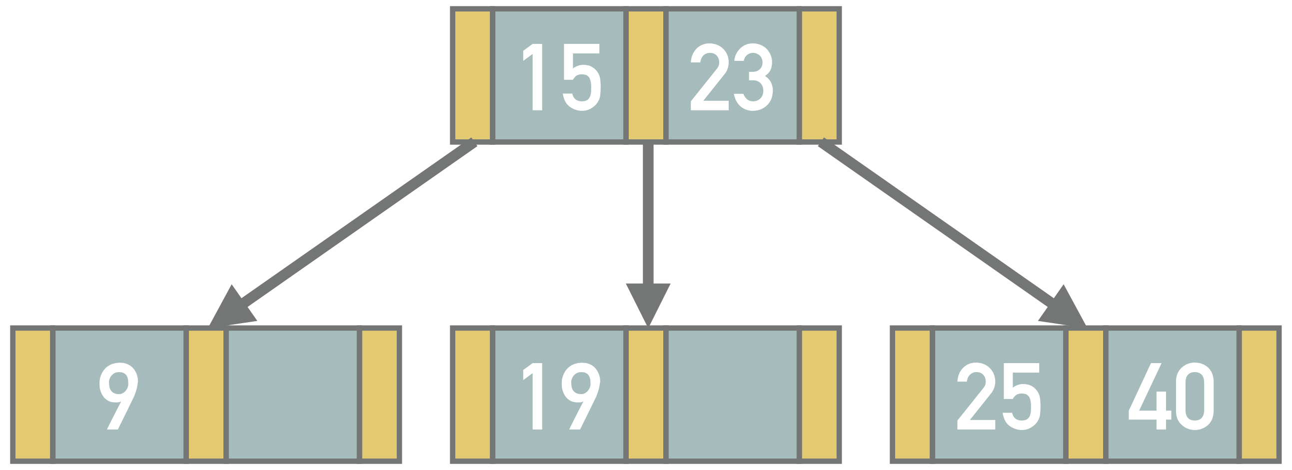
- Insert records with keys 25, 19 and 40 into the B-tree from previous example



- The record 25 fits into the (right) leaf



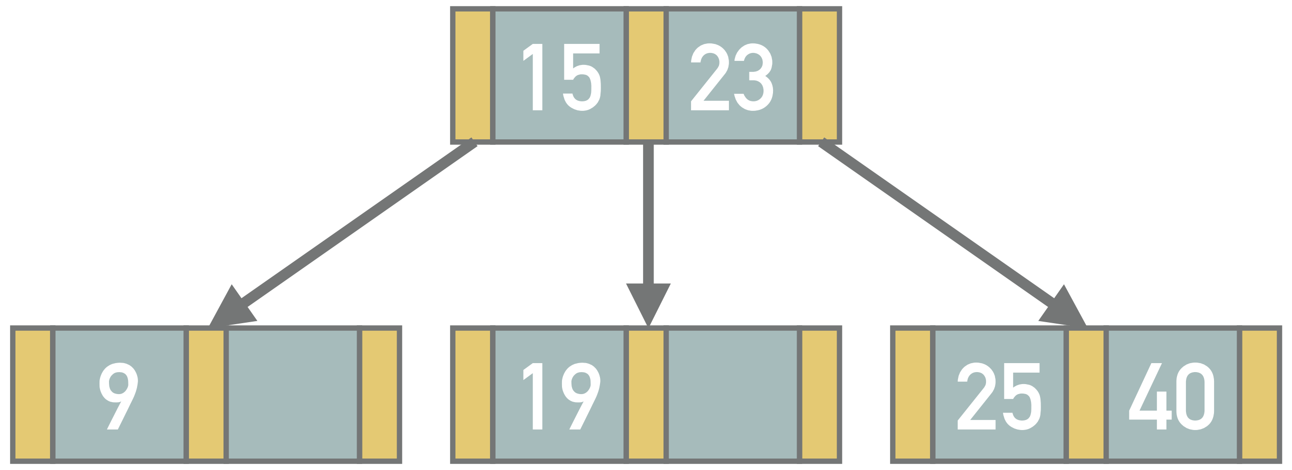
- The record with key 19 will split the (right) node into two nodes, i.e., (19) and (25) with (23) being the dividing record
  - The dividing record (23) finds its place in the parent node



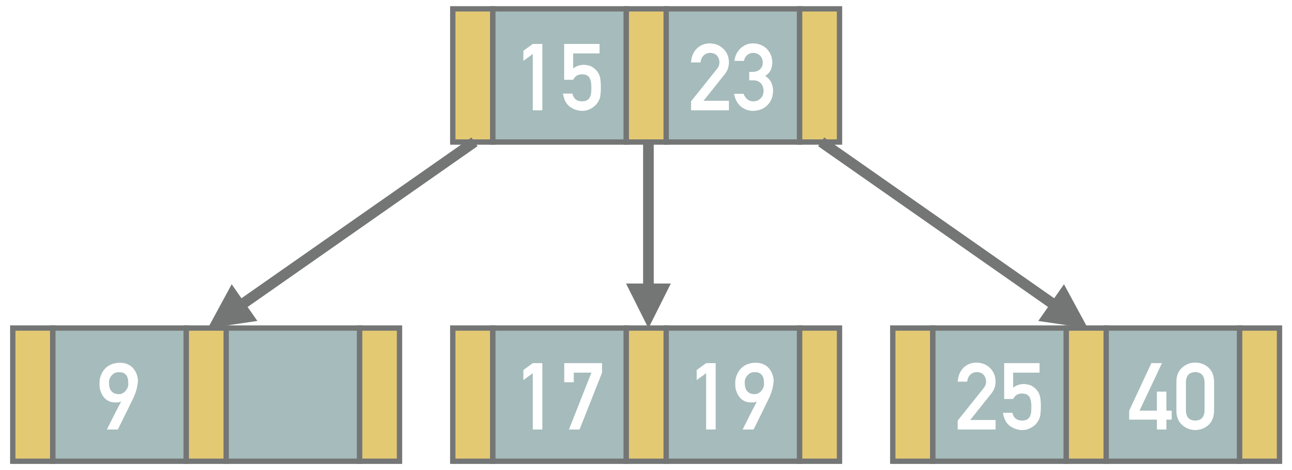
- The record 40 will fall into the right node

EXAMPLE 3: INSERT (PROPAGATION)

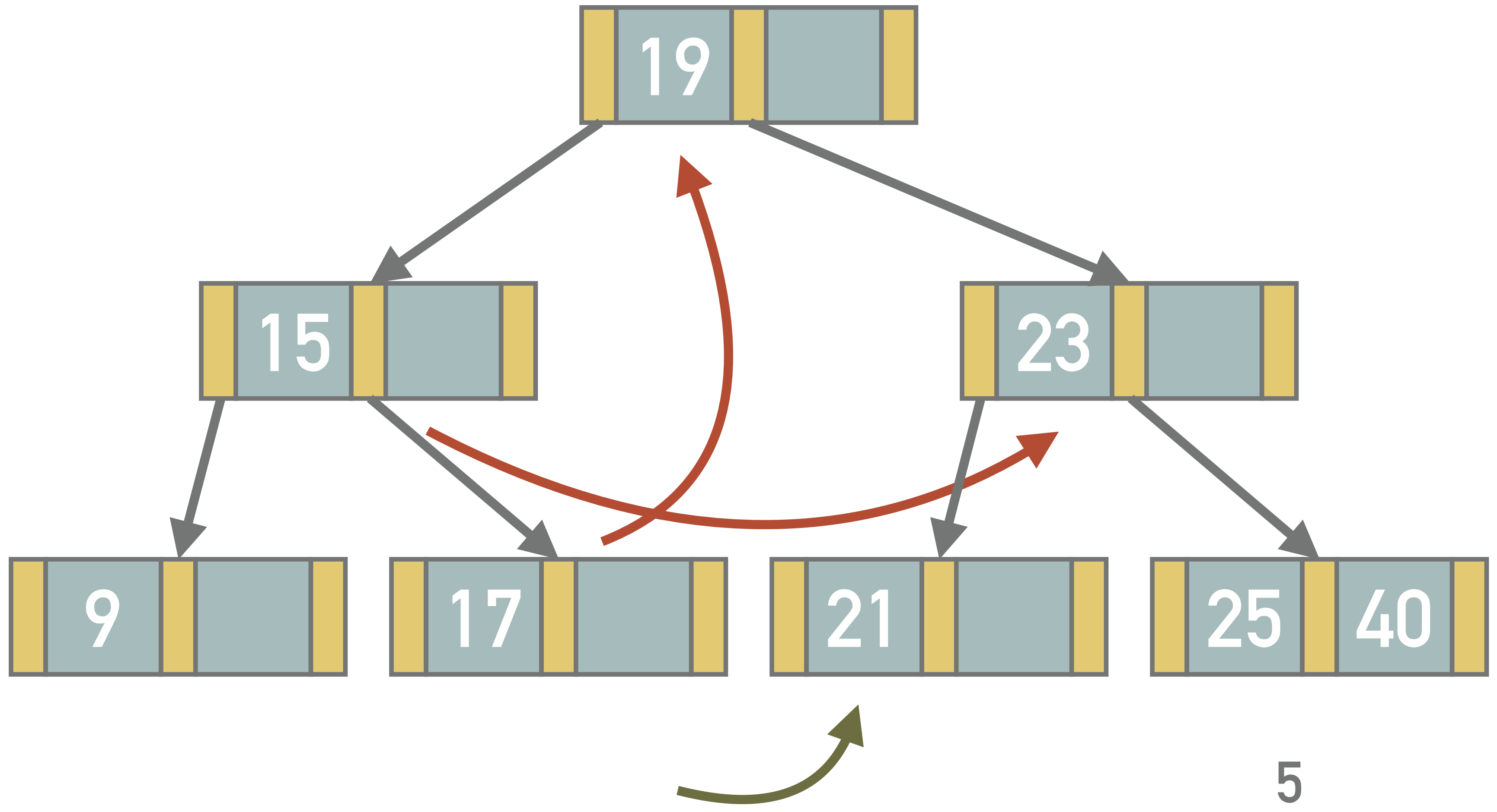
- Insert records with keys 17 and 21 into the B-tree from previous example



- The record 17 falls into the middle leaf

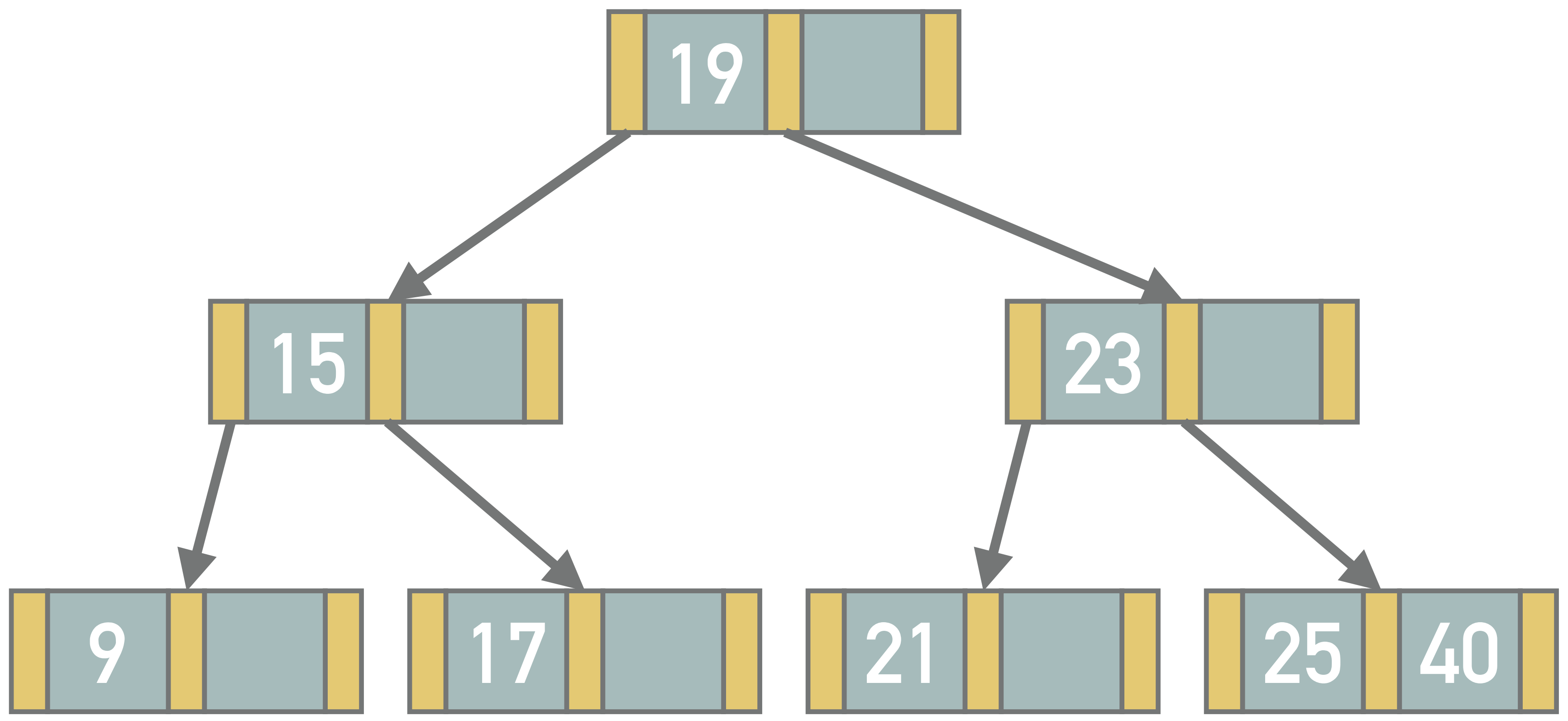


- The record 21 causes splitting of the middle leaf (17,19, 21) and propagation of the record (19) to the parent
  - However, there is no more space in the parent node (root)
  - Thus, the parent node(15,19,23) needs to be split as well which increases the tree height

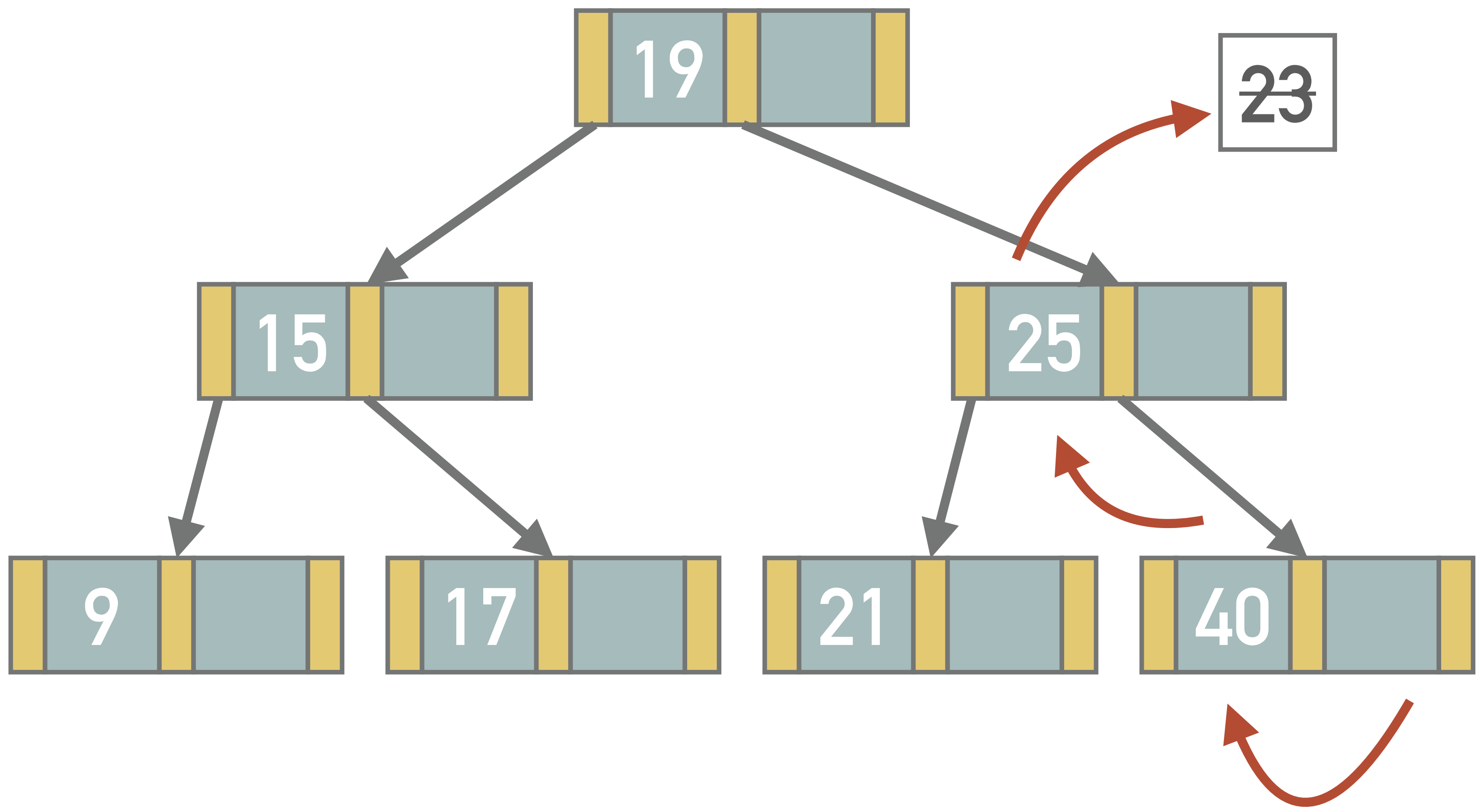


EXAMPLE 4: DELETE

- Remove record 23 from the non-redundant B-tree of degree 3 (see figure)

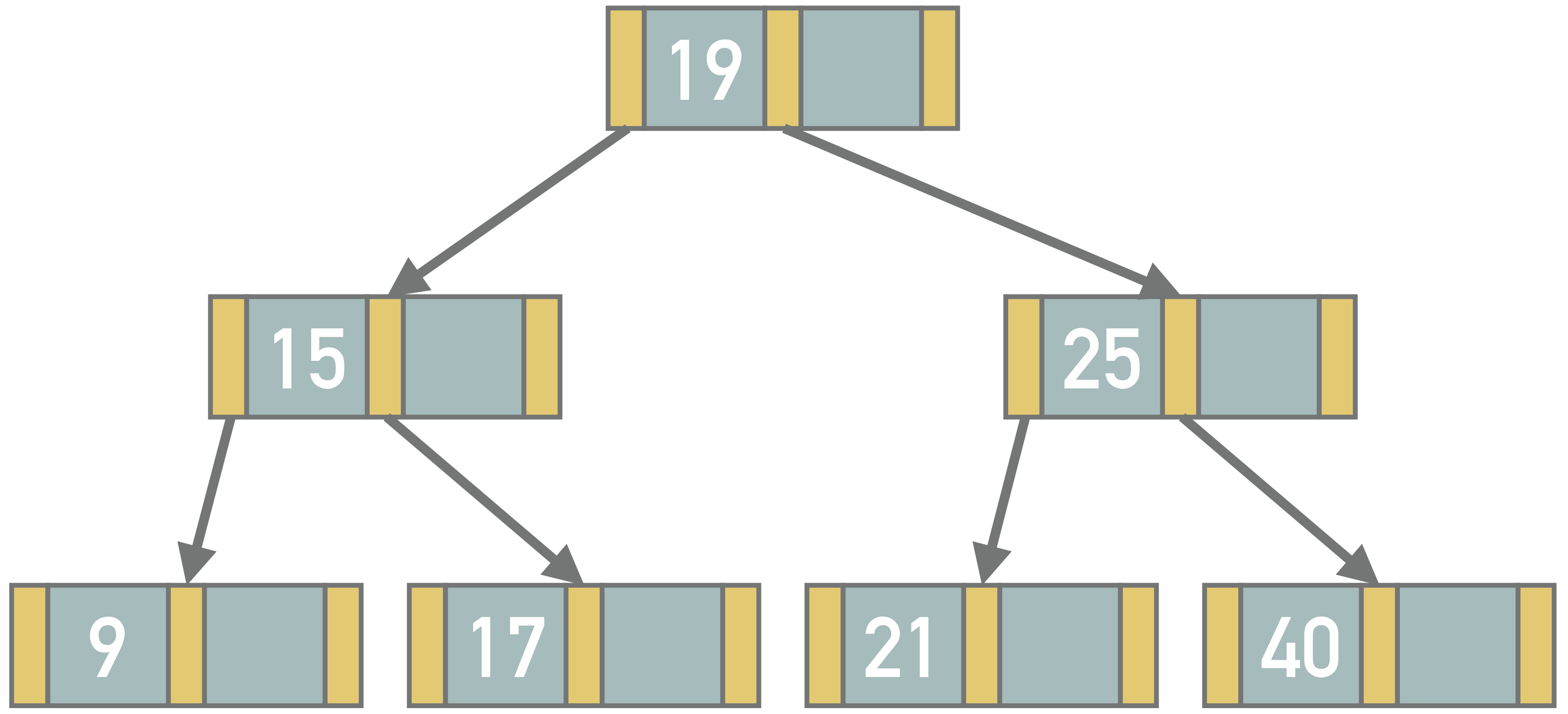


- The deletion of a data entry from an inner node leads to its replacement with the most left descendant entry from the right subtree or the most right entry from its left subtree
  - If we delete 23 from the tree above, we can replace it with entry 25 from the bottom node (leaf)
  - Moving the entry 25 from the leaf (25,40) is safe since it still has the minimum number of entries

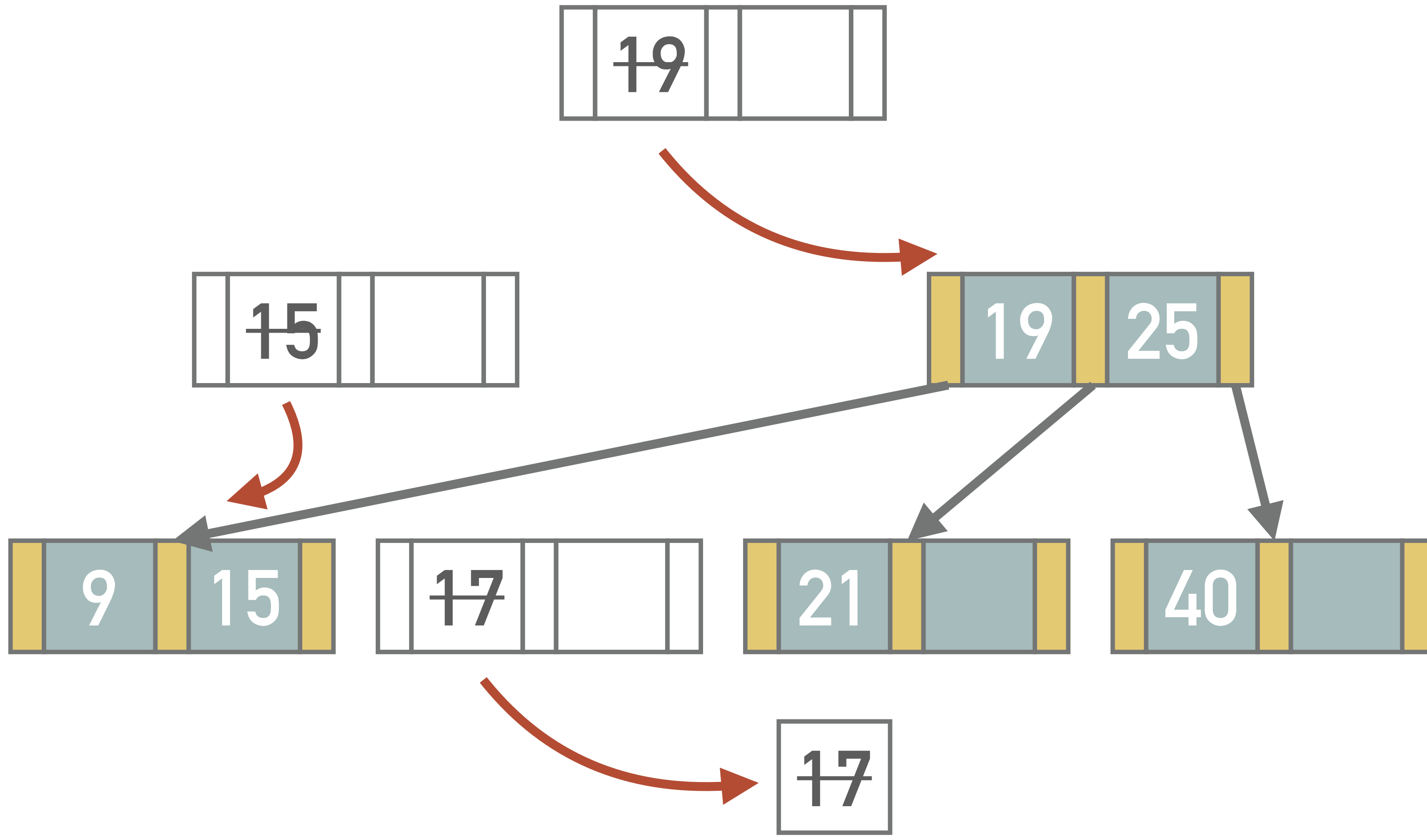


EXAMPLE 5: DELETE (MERGING)

- Remove record 17 from the non-redundant B-tree of degree 3 (see figure)



- We cannot borrow an entry from the neighbour (21) since it also contains the minimal number of entries
- We have to merge nodes (9), (empty) and (15)
  - The entries of the current node (none left after removing 17), those from the neighboring node (9) and the dividing entry will be moved into a single node (9,15)
    - Thus, the entry 15 needs to be removed from the parent node which causes underflow of that node
- We have to merge nodes (empty parent node), (19) and (25)
  - Once again, we cannot borrow an entry from the neighbour node (25)
  - The empty node () is merged with the node (25) and dividing entry (19) from the root node, resulting in the node (19,25)
  - Having entry 19 removed from root node (), the height of the tree decreases

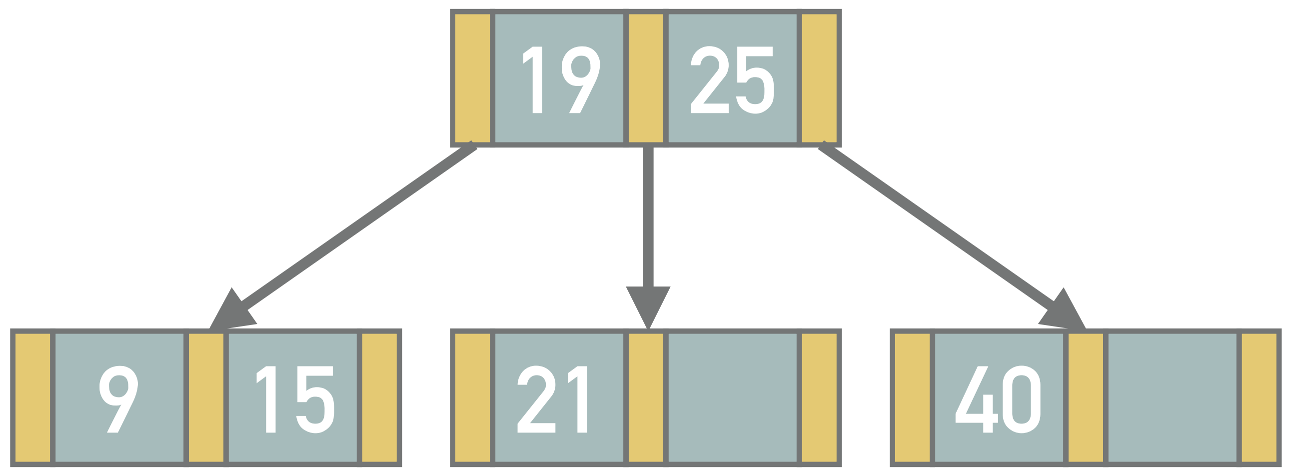


EXERCISE 1

- Suppose a non-redundant B-tree of degree $m = 3$ (see the figure)

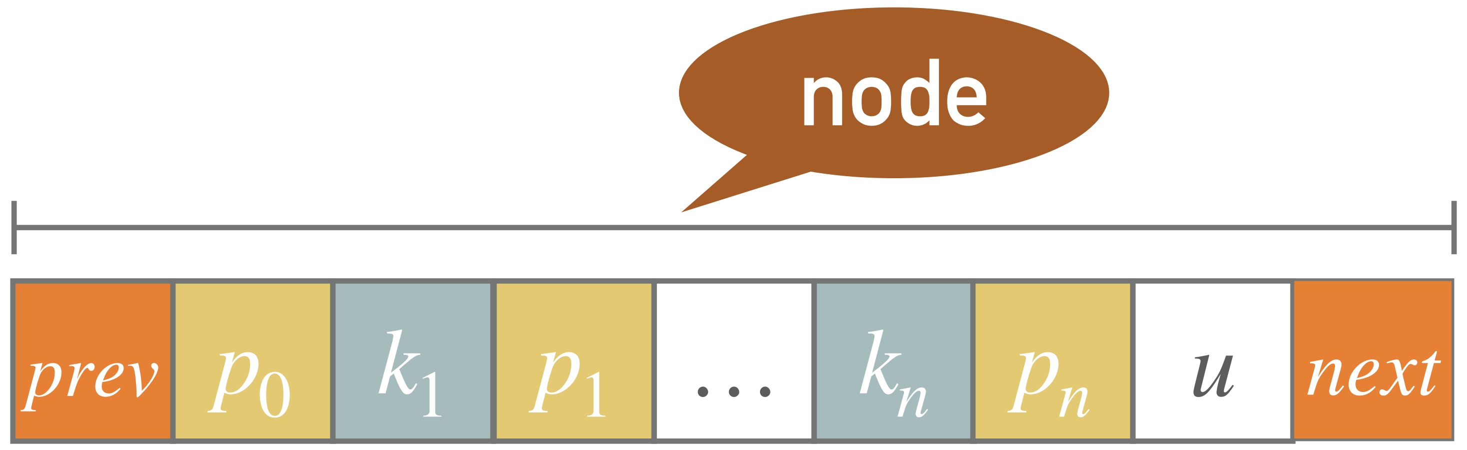
- First, illustrate the b-tree after insertion of records 11, 18 and 14

- Second, illustrate the b-tree after deletion of records 40 and 14



B+-TREE

- B+-Tree differs from the original B-tree by:
  - It is *always redundant*, i.e., the data are stored or pointed to from the leaf nodes
  - The *leaf nodes are chained* using pointers a linked list which simplifies range queries
    - In reality, often all the levels are linked (not just the leaf level)
  - The inner nodes contain only the values using which the tree can be traversed

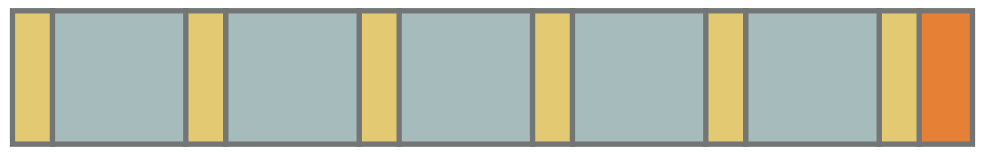


- The nodes have the structure $[prev,]\ p_0, (k_1, p_1), \ldots, (k_n, p_n), u[, next]$
- p_i - pointers to the children
- k_i - keys

- Keys k_j in the subtree pointed by p_i are greater than or equal to k_i and less than k_{i+1}, if k_{i+1} exists
- The minimum number of children can be raised to $\lceil (m+1)/2 \rceil$

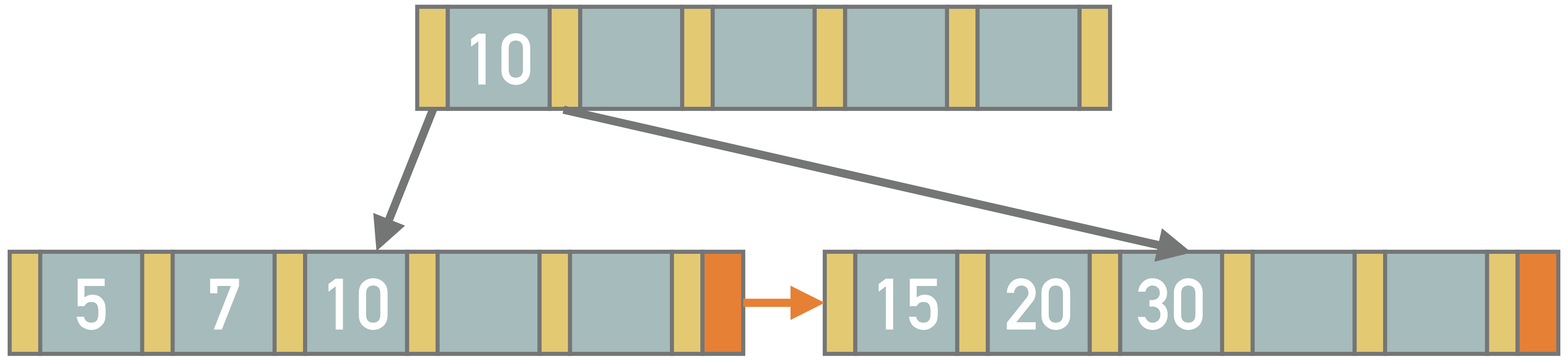
EXAMPLE 6: INSERT

- Insert records with keys 10, 7, 15, 5, 30 and 20 into an empty B+-tree
  - Suppose a B+-tree of degree $m = 6$
    - The minimum number of children is therefore 3



- Insertion of keys 10, 7, 15, 5 and 30 is trivial, all belong to the root node

- Insertion of key 20 leads to a page split
  - A half of the records, i.e., (5, 7, 10), stays in the original page while the rest, i.e., (15, 20, 30), moves into a new page
  - The max key value in the left node, i.e., 10, is propagated into the higher level (new root node)
    - However, any value $10 \leq value \leq 14$ would work



EXAMPLE 7: INSERT

- Insert additional records with keys 13, 3, 11, 21, 8 and 9 into the B^+-tree from previous example

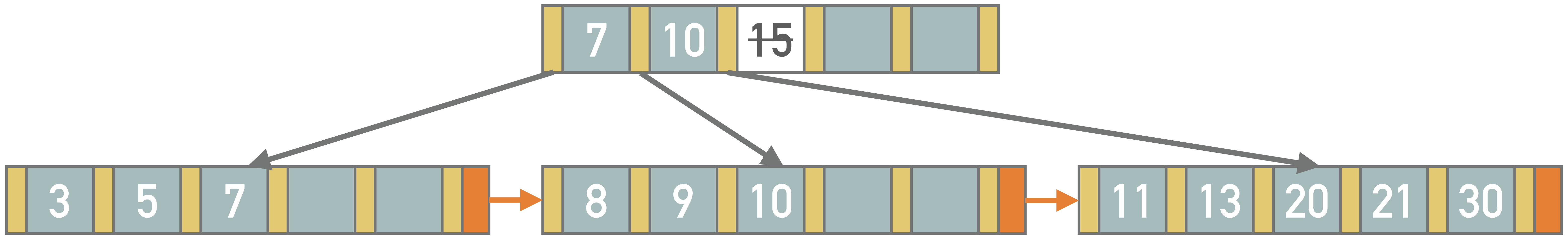
- The insertion of records with keys 13, 3 and 11 is trivial
- The insertion of a record with key 21 splits the right leaf node into nodes (11, 13, 15) and (20, 21, 30)
  - The separating value 15 is inserted into the parent node where there is enough space so it does not lead to another split
- Inserting of records with keys 8 and 9 leads to the split of the leaf into (3,5,7) and (8,9,10)
  - The separation value 7 is inserted into the parent node

EXAMPLE 8: DELETE (MERGE NODES)

- Remove the entry with key 15 from the B^+-tree (see the previous page)

- When removing entries from a B^+-tree, the given entry is simply removed from the leaf unless the corresponding leaf underflows
  - In such case, the tree tries to borrow an entry from a neighbouring leaf node (and to change the splitting value in the parent)
  - If also the neighbours have the minimum number of entries, it is necessary to merge two nodes into one and remove the splitting value from the parent
    - Which can lead to the merge cascade up to the root

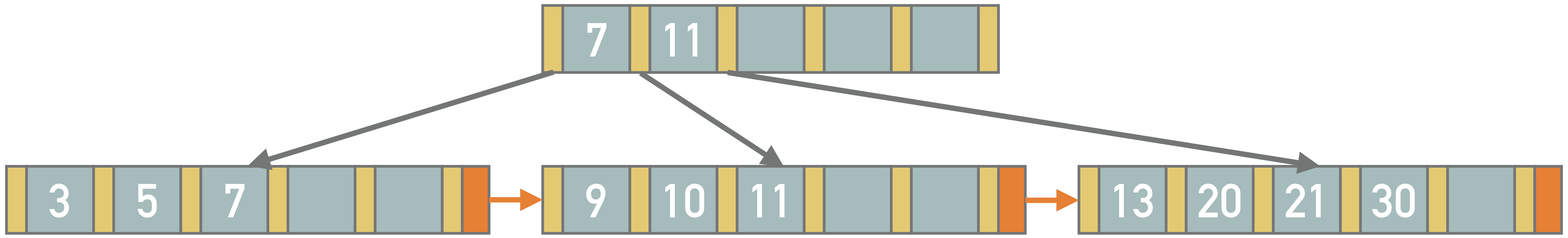
- In our example, every node (except the root) needs to include at least three keys
  - By removing the entry 15, this condition is violated and the neighbouring nodes cannot lose any entry either
  - Thus we merge node (11, 13) with (20, 21, 30) and remove the splitting value 15 from the parent



EXAMPLE 9: DELETE (BORROW KEY)

- Remove the entry with key 10 from the B+-tree (see the previous page)

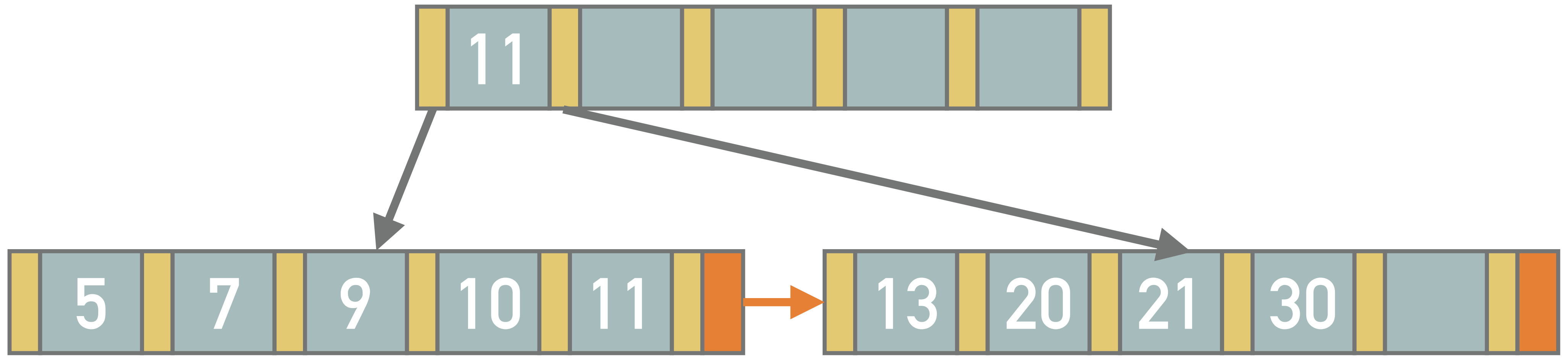
- To remove the entry 10 we need to move the entry with key 11 from the neighbouring node to keep the condition of minimum number of entries in every node
  - It is necessary to change the splitting value in the parent from 10 to 11



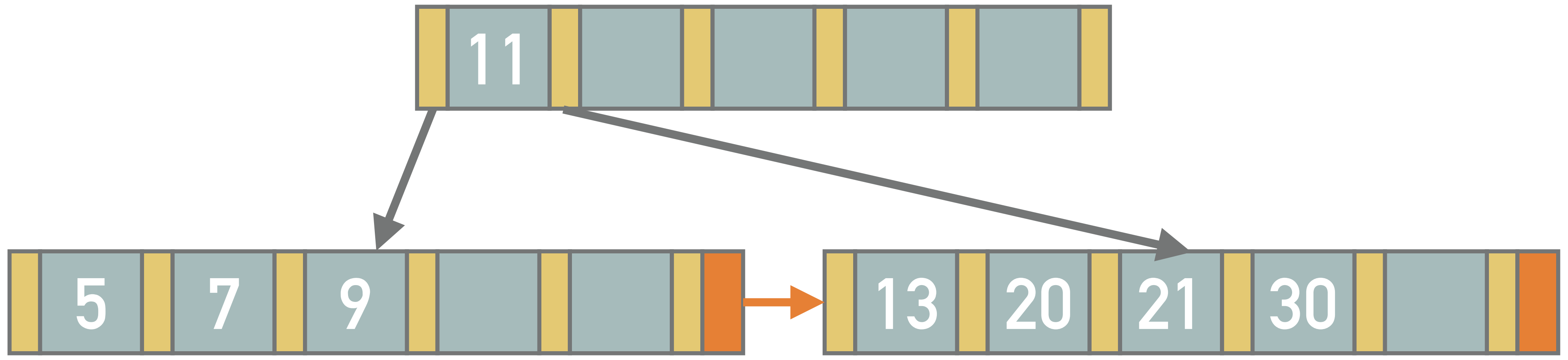
EXAMPLE 10: DELETE

- Remove records with keys 3, 10 and 11 from the B+-tree (see the previous page)

- Removing the key 3
  - After the removal, the number of records in the node (5, 7) falls under minimum and the neighbouring nodes, i.e., (9,10,11), cannot provide any record
  - The nodes (5,7) and (9, 10, 11) are merged
  - Finally, the splitting value 7 is removed from the parent



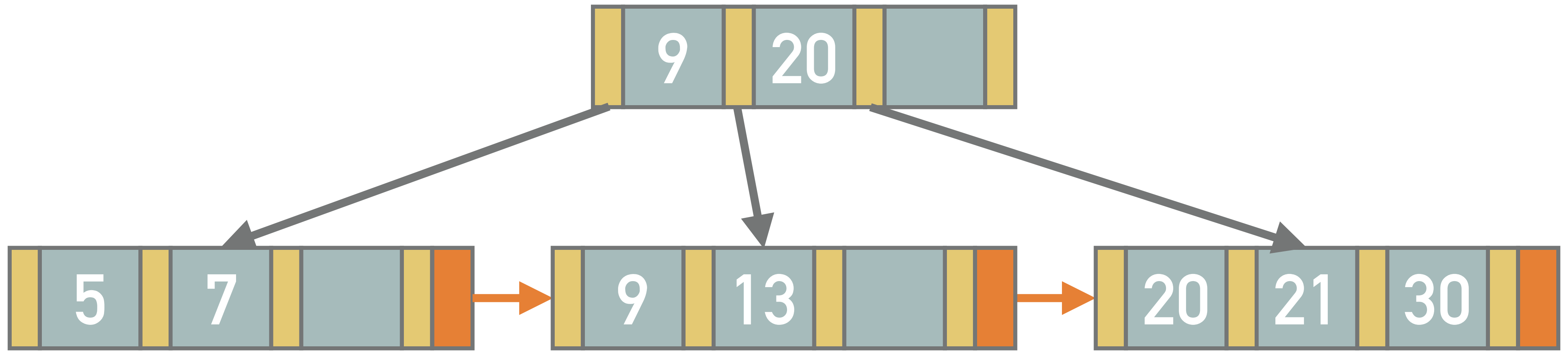
- Removing the keys 10, 11
  - It is sufficient to remove the keys from the node, no modifying of splitting value is needed



EXERCISE 2

- Suppose a B^+-tree of degree $m = 4$ (see the figure)
  - Minimum number of children in a node is 2

- Illustrate the B^+-tree after the insertion of records 40, 50 and 60

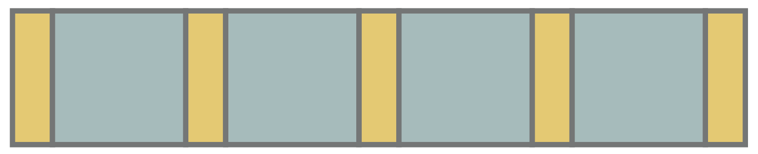


B*-TREE

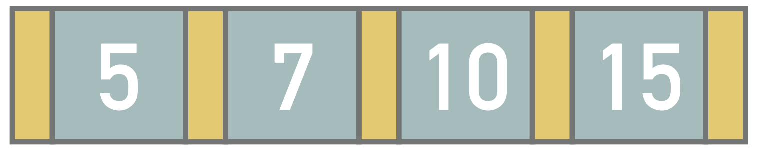
- B*-tree differ from the standard B-tree by:
  - The non-root nodes have at least $\lceil (2m-1)/2 \rceil$ children
    - If the tree contains few records (i.e., after splitting the root node), the only two leafs can contain less records (about half))
  - If a node has too few items, or overflows, it is balanced using both of its neighbours
  - If a node and its neighbour are full, they are split (together with the new record) into three nodes being 2/3 filled

EXAMPLE 11: INSERT

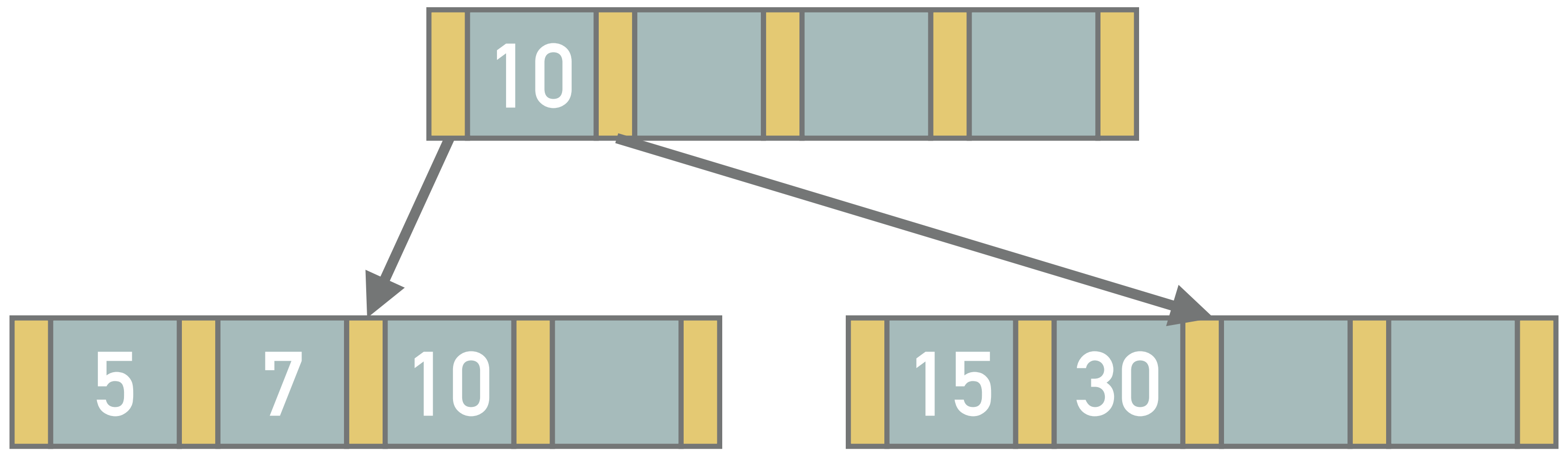
- Insert records with keys 10, 7, 15, 5, 30, 20 and 13 into an empty redundant B*-tree



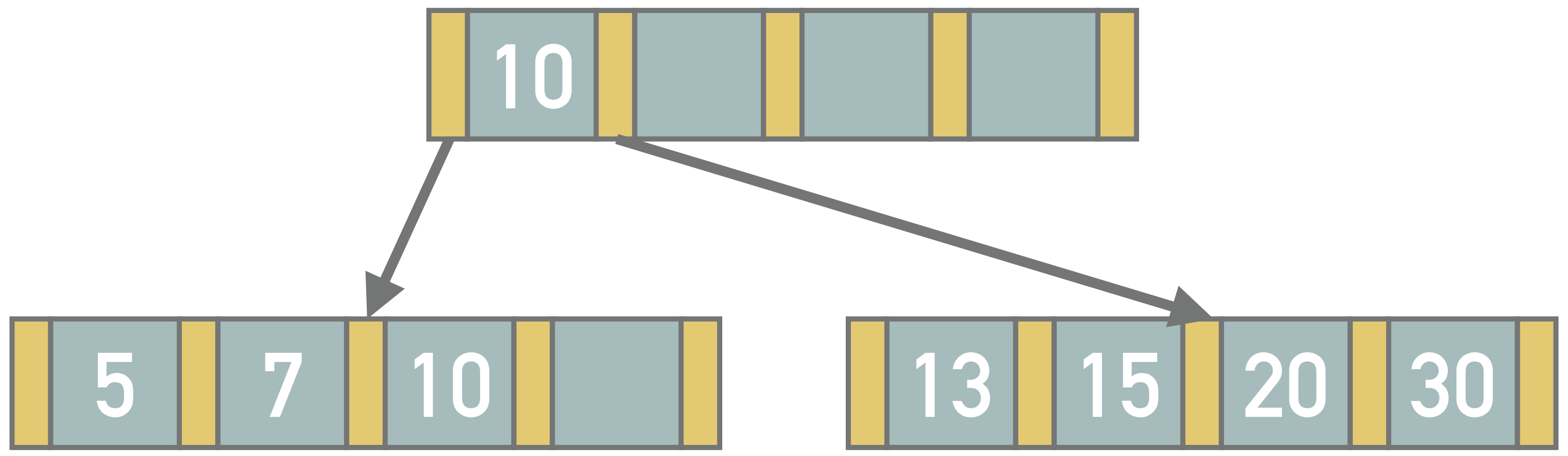
  - Suppose an empty B*-tree of degree $m = 5$
  - The minimum number of children is $\lceil \frac{2}{3}(m - 1) \rceil + 1 = \lceil \frac{2m - 2}{3} \rceil + 1 = 4$



- Insertion of records with keys 10, 7, 15 and 5 is trivial, all goes to the root node
- Inserting a record with key 30 leads to root node split
  - Split nodes are (5, 7, 10) and (15, 30)
  - The dividing value 10 is inserted into the new parent (new root)

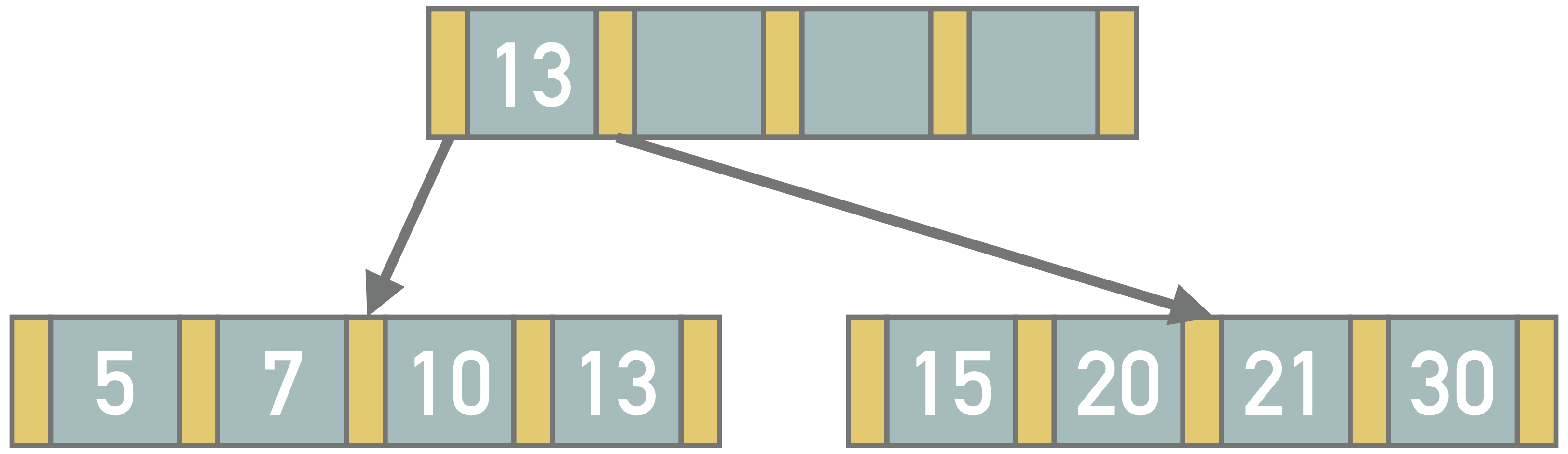


- A record with key 20 can be inserted into the right leaf, as well as a record with a key 13

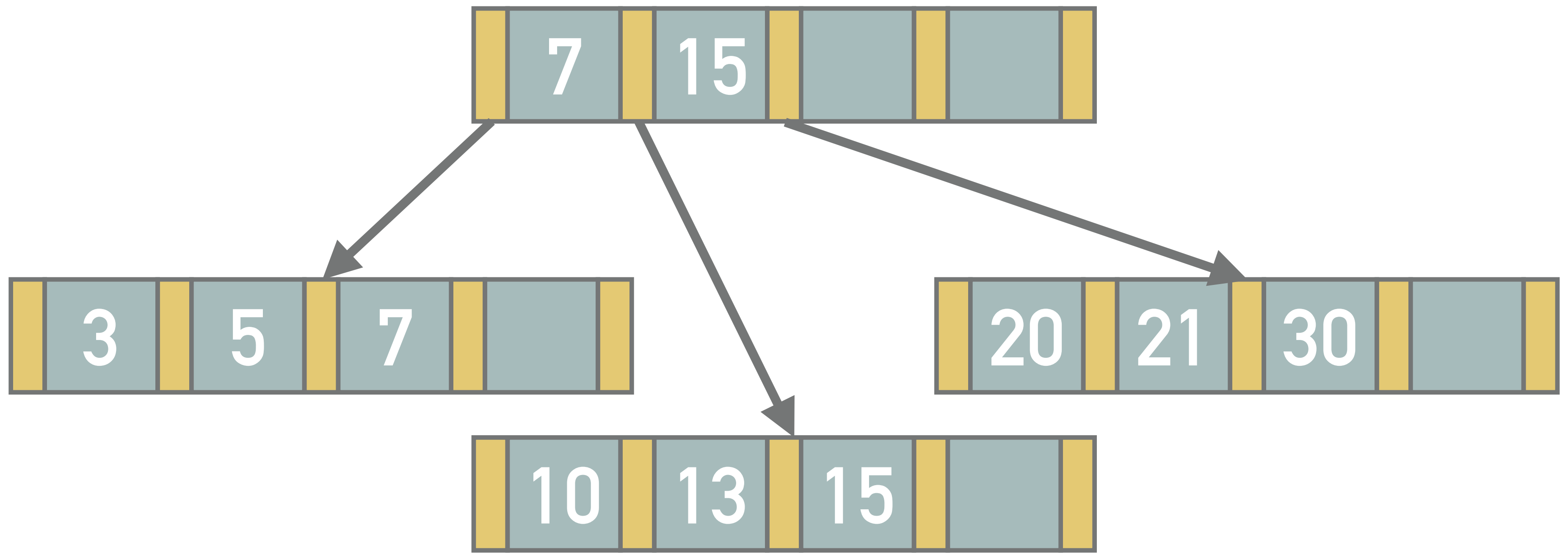


EXAMPLE 12: INSERT

- Continue with previous example and insert records with keys 21 and 3 into the redundant B*-tree



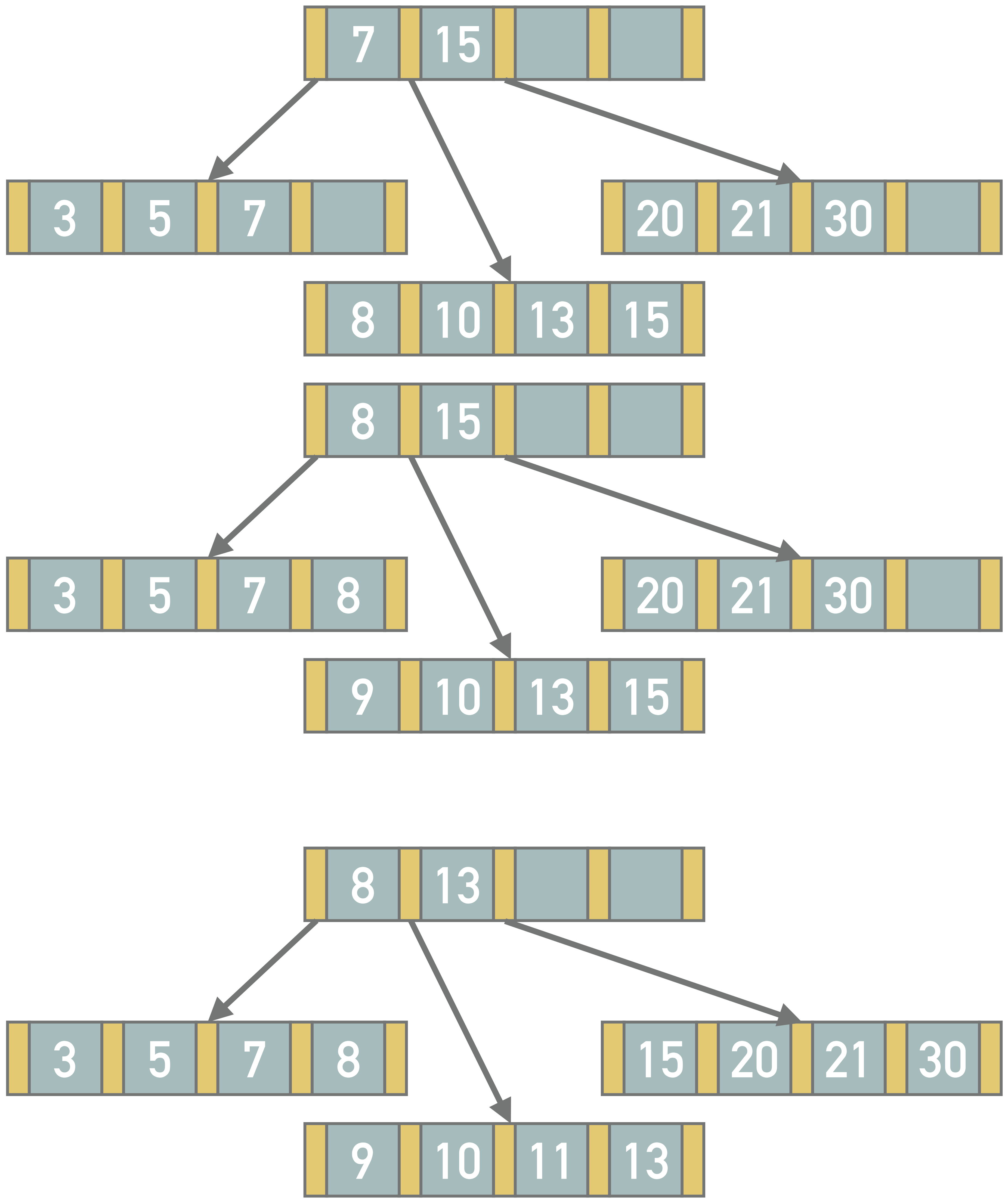
- Inserting the key 21
  - We cannot insert the key 21 into the full node (13, 15, 20, 30), but the record with key 13 can be moved to the neighbouring, not yet filled node
  - The splitting value in the parent needs to be modified



- Inserting the key 3
  - The key 3 cannot be inserted into the node (5, 7, 10, 13) and the neighbour is full as well
  - The records in both nodes, together with record 3, will be split into three nodes (3,5,7), (10, 13, 15) and (20, 21, 30)
  - Splitting values 7 and 15 need to be inserted into the parent node instead of the existing splitting value 13

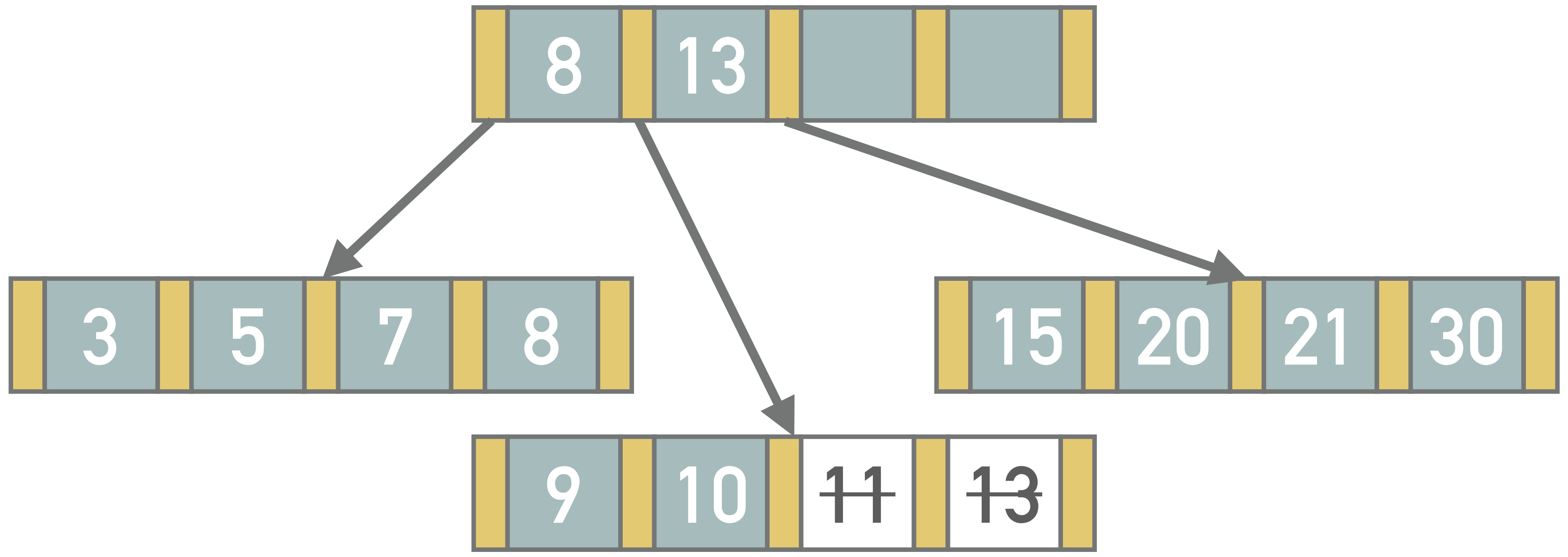
EXAMPLE 13: INSERT

- Continue with previous example and insert records with keys 8, 9 and 11 into the redundant B*-tree
- The record 8 fits into the middle leaf
- The record 9 causes redistribution of the record 8 to the left and change of the splitting value from 7 to 8
- The record with a key 11 will cause one of two possibilities:
  - **The redistribution of the record with key 15 to the right and modification of the splitting value in the parent from 15 to 13**
  - Split of nodes (3,5,7,8) and (9,10,13,15) into three nodes (3,5,7), (8,9,10) and (11,13,15)
    - The splitting value 8 would be replaced by a pair 7 and 10

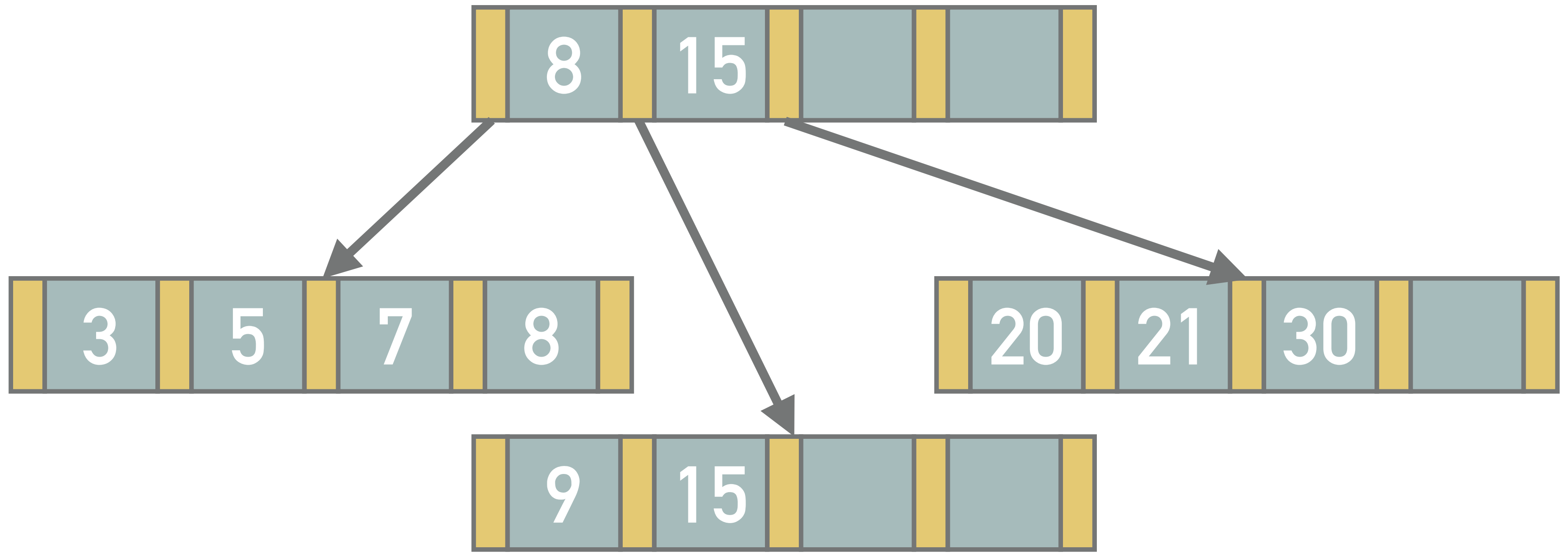


EXAMPLE 14: DELETE

- Continue with previous example and delete the records with keys 13, 11 and 10 from redundant B*-tree



- The record with key 13 can be easily deleted from the middle leaf
- The same holds for the record with key 11
- The record with key 10 cannot be deleted directly
  - The number of entries in a node would decrease under the threshold
  - Therefore it is necessary to mode there the record with key 15 from the neighbouring node
  - The splitting value in the parent changes from 13 to 15



EXERCISE 3

- Continue with previous example and delete the records with keys 15, 9 and 8 from redundant B*-tree
- Finally, remove (single) additional key of your choice from the B*-tree
  - Illustrate and comment the removals step by step