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SERIES AND TOEPLITZ MATRICES (A GLOBAL IMPLICIT APPROACH)

PETER VOJTÁŠ

Dedicated to Professor T. Šalát on the occasion of his 70th birthday

ABSTRACT. In this paper we deal with properties of series and Toeplitz matrices from set-theoretic point of view. This paper is motivated by results of N. N. Kholshchevnikova, V. I. Malykhin and T. Šalát. We consider cardinal characteristics of series and matrices (which usually lie between $\aleph_1$ and $2^{\aleph_0}$). So far this is no more interesting under CH, hence we reformulate all this in the language of Galois-Tukey connections between binary relations characterizing properties of series, sequences and matrices. To avoid further trivialities and to preserve forcing properties we consider Borel connections. We survey some of our former results in a new setting of Galois-Tukey connections and give also some new results and formulate several problems.

1. Motivation and introduction

Our motivation comes from the real analysis and set theory. First some classical results:

(1) For every sequence of real numbers there is a subsequence which converges (Weierstrass).

(2) For every absolutely convergent series there is one which converges asymptotically slowlier (du Bois-Reymond).

(3) For every bounded sequence of reals there is a matrix (now called Toeplitz) under which the very sequence converges or is summed by (Toeplitz).

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People in real analysis have already asked whether we can do this simultaneously for more than one series, sequence or matrix:

(4) For countably many absolutely convergent series there is one which converges asymptotically slowlier than all of them (F i c h t e n g o l z [Fi]).

(5) For countably many $0-1$ sequences there is a single matrix summing them all and no finite (countable) family of matrices covers all $0-1$ sequences (C o o k e [Co]).

(6) For countably many sequences of reals there is a single set of indices so that all corresponding subsequences converge (K h o l s h c h e v n i k o v a, M a l y k h i n [MKh]).

We translate all these observations to cardinal characteristics lying between $\aleph_1$ and $2^{\aleph_0} = \mathfrak{c}$, hence under CH they are equal to continuum $\mathfrak{c} = \omega_1$.

N. N. K h o l s h c h e v n i k o v a and V. I. M a l y k h i n have first shown that the question whether these cardinal characteristics equal the continuum is independent on axioms of set theory with negation of CH. We refine these results and add something more.

Our notation is standard set-theoretic one (see, e.g., [BJ]), the main difference for analysists probably is that ω denotes the set of natural numbers. Instead of ${}^{\omega}\omega$ sometimes we shall prefer to work with the set Mon $\subseteq {}^{\omega}\omega$ of strictly increasing functions (because the relations $\leq^* \cap \mathrm{Mon}$ and $\leq^*$ are Galois-Tukey equivalent).

As we already mentioned, we understand properties of series and matrices as properties of certain binary relations. Assume R is a binary relation such that $\mathrm{dom}(R) = \mathrm{dom}\big((\mathrm{dom}(R) \times \mathrm{rng}(R)) \setminus R\big)$ and $\mathrm{rng}(R) = \mathrm{rng}\big((\mathrm{dom}(R) \times \mathrm{rng}(R)) \setminus R\big)$. A set $B \subseteq \mathrm{dom}(R)$ is said to be R-unbounded if $\big(\forall y \in \mathrm{rng}(R)\big)$ $(\exists b \in B)$ $\big((b,y) \notin R\big)$. The bounding number $\mathfrak{b}(R)$ of the relation R is defined to be the minimal size of an R-unbounded set. A set $D \subseteq \mathrm{rng}(R)$ is said to be R-dominating if $\big(\forall x \in \mathrm{dom}(R)\big)$ $(\exists d \in D)$ $\big((x,d) \in R\big)$. The dominating number $\mathfrak{d}(R)$ of the relation R is defined to be the minimal size of an R-dominating set.

Note that $\mathfrak{b}(R) = \mathfrak{d}(\neg R^{-1})$ and $\mathfrak{d}(R) = \mathfrak{b}(\neg R^{-1})$. Later we show that cardinals involved in (4), (5), (6) are of this sort.

DEFINITION 1 [Vo3]. For two binary relations R and S we say that a couple (E,F) is a *Galois-Tukey connection* from R to S if

(i) $E\colon \mathrm{dom}(R) \to \mathrm{dom}(S)$,

(ii) $F\colon \mathrm{rng}(S) \to \mathrm{rng}(R)$,

(iii) $\big(\forall x \in \mathrm{dom}(R)\big)$ $\big(\forall v \in \mathrm{rng}(S)\big)$ $\Big(\big(E(x), v\big) \in S \Rightarrow \big(x, \big(F(v)\big) \in R\big)\Big)$.

If there is such a couple we denote this situation by $R \to S$ or by $R \leq S$ and say that R is *simpler* than S.

OBSERVATION 2.

(i) $R \to S$ *if and only if* $\neg S^{-1} \to \neg R^{-1}$.

(ii) *If* $R \to S$, *then* $\mathfrak{b}(S) \leq \mathfrak{b}(R)$ *and* $\mathfrak{d}(R) \leq \mathfrak{d}(S)$.

Proof. For (ii) just note that an E-image of an R-unbounded set is S-unbounded and F-image of an S-dominating set is R-dominating. □

This is a generalization of Tukey mappings introduced by Fremlin ([Fr1, 2]) and in a special case by Pawlikowski (see [BJ]) for partial orders. We have generalized it to arbitrary binary relations in [Vo3]. The statement (ii) of the observation shows that in order to prove inequalities between cardinal characteristics it suffices to prove the implication involved in (iii) of Galois-Tukey connection (and of course to construct E and F). We hoped that this is interesting also under CH. This hope was destroyed by the following

OBSERVATION 3 (Yiparaki [Yi]). *If* $\mathfrak{b}(R) = |\mathrm{dom}(R)| = \mathfrak{d}(S) = |\mathrm{rng}(S)| = \kappa$, *then there is a Galois-Tukey connection from* R *to* S.

Proof. Enumerate $\mathrm{dom}(R) = \{x_\alpha : \alpha < \kappa\}$ and $\mathrm{rng}(S) = \{v_\alpha : \alpha < \kappa\}$ and define $E(x_\alpha)$ to be a witness that $\{v_\beta : \beta < \alpha\}$ is not S-dominating and $F(v_\alpha)$ be a witness that $\{x_\beta : \beta < \alpha\}$ is not R-unbounded. Then $(E(x_\gamma), v_\delta) \in S$ implies $\gamma \geq \delta$, hence $(x_\gamma, F(v_\delta)) \in R$. □

As far as all relations concerned in real analysis have size $\mathfrak{c}$, this shows that under CH, between any two relations with uncountable $\mathfrak{b}$'s and $\mathfrak{d}$'s there is a Galois-Tukey connection.

In [Vo3] we have studied connections for which E and F are Borel only because of preservation of forcing properties. After Yiparaki's result it turned out that the study of Borel connections is moreover a reasonable alternative which is interesting also under CH (see [Bl1, Bl2, Mi]).

2. Series

Now we define relations of being convergent/divergent asymptotically faster. In this paper we consider absolute convergence of series, hence we deal with

$$\ell^1_+ = \ell^1 \cap {}^\omega(0,+\infty) \qquad \text{and} \qquad \ell^\infty_+ = \ell^\infty \cap {}^\omega(0,+\infty).$$

Let $\mathrm{CAF} \subseteq \ell^1_+ \times \ell^1_+$ be defined as follows

$$(a,b) \in \mathrm{CAF} \qquad \text{iff} \qquad \lim_{n\to\infty} \frac{\sum\limits_{k=n}^{\infty} a_k}{\sum\limits_{k=n}^{\infty} b_k} = 0$$

(this is a limit of type $\frac{0}{0}$, hence a_n's are in nominator, we read this: the series a Converges Asymptotically Faster than the series b). Let $\mathrm{DAF} \subseteq \ell_+^\infty \times \ell_+^\infty$ be defined as follows

$$(a, b) \in \mathrm{DAF} \qquad \text{iff} \qquad \lim_{n\to\infty} \frac{\sum_{k=0}^{n} b_k}{\sum_{k=0}^{n} a_k} = 0$$

(this is a limit of type $\frac{\infty}{\infty}$, hence a_n's are in denominator, we read this: the series a Diverges Asymptotically Faster than the series b).

The Fichtengolz observation can now be reformulated as

$$\aleph_1 \le \mathfrak{b}(\mathrm{CAF}) \le \mathfrak{c} \qquad \text{and} \qquad \aleph_1 \le \mathfrak{b}(\mathrm{DAF}) \le \mathfrak{c}\,.$$

N. N. Kholshchevnikova in [Kh] has shown that:

THEOREM 4 [Kh]. *Both* $\mathfrak{b}(\mathrm{CAF}) = \mathfrak{c}$ *and* $\mathfrak{b}(\mathrm{DAF}) = \mathfrak{c}$ *are independent of set theory with the negation of CH.*

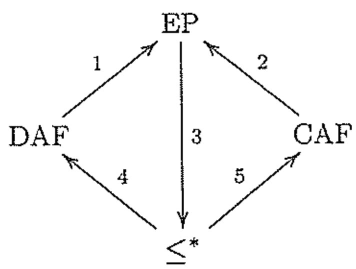
The combinatorial content of Kholshchevnikova's proof gives in fact a Galois-Tukey connection from DAF and CAF into something we call Erdös–Piranian relation ([EP]).

DEFINITION 5. Let $\mathrm{EP} \subseteq \mathrm{Mon} \times [\omega]^\omega$ be defined as follows

$$(f, X) \in \mathrm{EP} \quad \text{if} \quad \forall^\infty n \ \big|[f(n), f(n+1)) \cap X\big| \le 1\,.$$

Recall that for $f, g \in {}^\omega\omega$, $f \le^* g$ iff $\forall^\infty n\ f(n) \le g(n)$ (here we restrict ourselves to strictly increasing functions).

THEOREM 6.



Hence all relations are Galois-Tukey equivalent (due to Borel morphisms).

Proof. The connections $\xrightarrow{1}$ and $\xrightarrow{2}$ are constructed in [Kh].

$\xrightarrow{3}$. The proof involves an old trick due independently to P. Simon, P. Nyikos, and others (see [Va]). To show $\mathrm{EP} \longrightarrow \le^*$ we have to define

E_3: Mon $\to$ Mon and F_3: Mon $\to [\omega]^\omega$ fulfilling (iii) of definition of Galois-Tukey connections. Take $E_3(f) = f$ and to define $F_3(g)$ first define $\overline{g}(0) = 0$ and $\overline{g}(n+1) = g(\overline{g}(n)+1)$ and then put $F_3(g) = \operatorname{rng}(\overline{g})$. Then whenever $f \leq^* g$ then (for sufficiently large i and n) $f(i) \leq \overline{g}(n) < f(i+1)$ implies $\overline{g}(n+1) = g(\overline{g}(n)+1) \geq f(\overline{g}(n)+1) \geq f(f(i)+1) \geq f(i+1)$, hence $\big|[f(i), f(i+1)) \cap \operatorname{rng}(\overline{g})\big| \leq 1$ and $(f, F_3(g)) \in \mathrm{EP}$.

$\xrightarrow{4}$. For an $f \in$ Mon assign $E_4(f)(i) = 0$ for $i \in [0, f(0))$ and

$$E_4(f)(i) = \frac{1}{f(k+1) - f(k)} \qquad \text{for} \quad i \in [f(k), f(k+1))$$

(hence $E_4(f) \in \ell^\infty_+ \setminus \ell^1_+$). For and $b \in \ell^\infty_+ \setminus \ell^1_+$ set

$$F_4(b)(n) = \min\Big\{j\colon \sum_{i=0}^{j} b_i > n\Big\}.$$

Assume that $(E_4(f), b) \in \mathrm{DAF}$. There is an n_0 such that

$$\sum_{k=0}^{n} b(k) < \sum_{k=0}^{n} E_4(f)(k)$$

for $n \geq n_0$ and then for some j_0 with $f(j_0) > n_0$ for all $j > j_0$ we have

$$j = \sum_{k=0}^{f(j)} E_4(f)(k) > \sum_{k=0}^{f(j)} b(k)$$

hence $f(j) < F_4(b)(j)$.

$\xrightarrow{5}$. This proof is similar to the previous one. Define

$$E_5(f)(i) = 2^{-k}/\big(f(k+1) - f(k)\big)$$

and

$$F_5(c)(k) = \min\Big\{j\colon \sum_{i=j}^{\infty} c_i < 2^{-k+1}\Big\} \qquad \text{for} \quad i \in [f(k), f(k+1)).$$

Again $(E_5(f), c) \in \mathrm{CAF}$ implies that for some n_0 and j_0 sufficiently large we have

$$\sum_{i=f(j)}^{\infty} E_5(f)(i) = 2^{-j+1} < \sum_{i=f(j)}^{\infty} c_i$$

and hence $F_5(c)(j) > f(j)$ for all $j > j_0$. □

So we see that asymptotical convergence/divergence is Borel Galois-Tukey equivalent to "the speed of approaching infinity by sequences of natural numbers" which are "scales of infinity" of du Bois-Reymond ([Ha]). The numbers $\mathfrak{b} = \mathfrak{b}(\leq^*)$ and $\mathfrak{d}(\leq^*)$ have been already intensively studied, hence the variety of models showing independence of $\mathfrak{b}(\mathrm{CAF}) = \mathfrak{b}(\mathrm{DAF}) = \mathfrak{c}$ equals those showing independence of $\mathfrak{b} = \mathfrak{c}$. Moreover, Erdös-Piranian hypothesis is equivalent to the statement $\mathfrak{b} = \mathfrak{c}$ and there is an effective way how to translate an $a \in \ell^1_+$ to an element of ℓ^∞_+ and vice versa showing that CAF and DAF, as partial orders, are Galois-Tukey equivalent.

There are another set-theoretic and/or global (e.g., topological) investigations of series. In [BEŠ] J. Belasová, J. Ewert and T. Šalát had shown that the sets of series the convergence of which can be decided by d'Alembert test (i.e., the ratio test), by Cauchy test (i.e., the root test), and by Raabe's ratio test, respectively, are all dense and of first Baire category in ℓ^1 (with the usual norm). We have generalized this result in [Vo5].

Another surprising result was T. Bartoszyński's ([Ba]) equality of the additivity of Lebesgue measure and $\mathfrak{b}(\leq^*_{\ell^1})$, i.e., the boundness number of ℓ^1 under eventual dominance (as in comparison test). Note that ℓ^1 under $\leq^*$ is upward directed.

The situation with absolutely divergent series ordered as in comparison test for divergence is quite different (in contrast to asymptotical properties we have shown to be equivalent). There are two ($\mathfrak{c}$ many) divergent series such that no divergent series is eventually below both of them. In [V7] we have shown that under $\mathfrak{p} = \mathrm{cf}(\mathfrak{c})$ the complete Boolean algebra generated by $(\ell^\infty_+ \setminus \ell^1_+, \leq^*)$ is isomorphic to the Boolean completion of $\mathcal{P}(\omega)/\mathrm{fin}$. The final ZFC problem was solved by S. Shelah ([Sh], [FSV]) showing that there are models of set-theory where these two algebras are not isomorphic. Hence a new structure (Boolean algebra) was born (although we used it implicitly for hundred years — just the global point of view brought it up) and it is challenging to study its properties. Newly T. Bartoszyński and M. Scheepers [BS] have shown that $\mathfrak{t}$-numbers of these two orderings are equal always (in ZFC).

3. Matrices

Mathematicians always wanted the sequence 0, 1, 0, 1, 0, 1, ... should have limit 1/2 and looked for methods to do so (e.g., Cesaro). The generalization of the notion of limit very often requires the method to be a linear functional prolongating the classical Bolzano-Weierstrass limit. We will study the phenomenon of limit as a relation between sequences (ℓ^∞, ${}^\omega 2$, ...) and the very method of calculation of some generalization of limit (subsequences, Toeplitz matrices, ...).

An infinite matrix of reals $A = \{a_{ij}: i, j \in \omega\}$ is said to be Toeplitz matrix (i's range through rows and j's range through columns) if

(i) $\exists m\ \forall^\infty i \quad \sum_{j=0}^{\infty} |a_{ij}| < m$,

(ii) $\lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij} = 1$,

(iii) $\forall j\ \lim_{\infty} a_{ij} = 0$.

Let $\mathscr{M}$ denote the set of all regular matrices. Recall that for $X \in [\omega]^\omega$ there is the unique function $e_X \in$ Mon such that $\operatorname{rng}(e_X) = X$ and $f = e_{\operatorname{rng}(f)}$ for $f \in$ Mon. For $f \in$ Mon let $D_f = \{d_{ij}: i, j \in \omega\}$ denote the diagonal matrix defined by $d_{ij} = 1$ if $j = f(i)$ and $d_{ij} = 0$ otherwise. Let $\mathscr{D}$ denotes the set of all diagonal matrices (which are sometimes identified with $[\omega]^\omega$).

For a matrix A and a sequence of reals $x = \{x_n\}_{n=0}^\infty$ we define the A-limit of x_n as the classical limit (if it exists) of the sequence $A \cdot x$ (obtained by a matrix multiplication where the sequence x_n is seen as a vertical vector), i.e.,

$$A\text{-}\lim_{n\to\infty} x_n = \lim A \cdot x = \lim_{i\to\infty} \sum_{j=0}^{\infty} a_{ij} x_j .$$

Note that, e.g., the Cesaro limit is a C-limit for $c_{ij} = 1/(i+1)$ if $j \le i$ and $c_{ij} = 0$ otherwise. Moreover $D_f\text{-}\lim_{n\to\infty} x_n = \lim_{i\to\infty} x_{f(i)}$. Remember A-limit is sometimes called a summation method, we say x_n is summed by A. In order to study global (implicit) properties of limit we need $\mathrm{LIM} \subseteq (\ell^\infty \setminus c) \times \mathscr{M}$ defined as $(x, A) \in \mathrm{LIM}$ if $A\text{-}\lim_{n\to\infty} x_n$ does exist. As far as we discuss different domains and ranges in order to have suitable notation we define $L_{\infty\mathscr{D}} = \mathrm{LIM} \cap \left[(\ell^\infty \setminus c) \times \mathscr{D}\right]$ (mnemonically $\infty\mathscr{D}$ resembles ℓ^∞ and $\mathscr{D}$), $L_{2\mathscr{D}} = \mathrm{LIM} \cap \left[({}^\omega 2 \setminus c) \times \mathscr{D}\right]$, $L_{\infty\mathscr{M}} = \mathrm{LIM}$, and $L_{2\mathscr{M}} = \mathrm{LIM} \cap \left[({}^\omega 2 \setminus c) \times \mathscr{M}\right]$. Corresponding negations of inverse relations (called mnemonically chaos, read χaos) we denote $\chi_{\mathscr{D}\infty}$, $\chi_{\mathscr{D}2}$, $\chi_{\mathscr{M}\infty}$, $\chi_{\mathscr{M}2}$ (note the order of method and sequences for chaos relations is opposite as for limit relations, namely methods are in the domain and sequences are in the range).

N. N. Kholschevnikova and V. I. Malykhin have shown:

THEOREM 7 [(MKh)]. *Both $\mathfrak{b}(L_{\infty\mathscr{M}}) = \mathfrak{c}$ and $\mathfrak{d}(L_{\infty\mathscr{M}}) = \mathfrak{c}$ are independent of set theory with the negation of CH.* □

If we take the combinatorial content of the proofs of [MKh], then they have shown there is a Galois-Tukey (Borel) connection from $\chi_{\mathscr{M}2}$ to EP (hence also from $\chi_{\mathscr{M}\infty}$ to $\le^*$, and hence $\mathfrak{b}(L_{2\mathscr{M}}) \le \mathfrak{d}$). Note that $\chi_{\mathscr{D}2} \to \le^*$ was already known as the combinatorial content of $\mathfrak{s} \le \mathfrak{d}$ (see [Va]), and moreover note that from pure inclusions we have

$$\begin{array}{ccc} \chi_{\mathscr{M}\infty} & \longrightarrow & \chi_{\mathscr{M}2} \\ \uparrow & & \uparrow \\ \chi_{\mathscr{D}\infty} & \longrightarrow & \chi_{\mathscr{D}2} \end{array} \qquad \begin{array}{ccc} L_{2\mathscr{D}} & \longrightarrow & L_{\infty\mathscr{D}} \\ \uparrow & & \uparrow \\ L_{2\mathscr{M}} & \longrightarrow & L_{\infty\mathscr{M}} \end{array}$$

One consistency in [MKh] shows that in an extension by ω_1-many Cohen reals there are ω_1 many $0-1$ sequences such that for arbitrary matrix A there are at most countably many of those sequences summed by A.

In [Vo3] we have already shown the connection of properties of Cohen extensions and Borel Galois-Tukey connections from $\in \bigcap([0,1] \times \mathscr{K})$. The following holds

THEOREM [(Vo1,2)]. *$\chi_{\mathscr{M}2} \to \not\ni \bigcap(\mathscr{K} \times [0,1])$, and hence $\mathfrak{b}(L_{2\mathscr{M}}) \le \mathrm{non}(\mathscr{K})$.*

Proof. In [Vo1] it was shown that for a matrix A the set $E(A) = \{x \in {}^{\omega}2\colon A\text{–}\lim x_n \text{ exists}\}$ is of first category. Identifying $[0,1]$ and ${}^{\omega}2$ and putting $F = \mathrm{id}$ we have established the connection (by Borel mappings). □

Hence every model of $\mathrm{non}(\mathscr{K}) = \omega_1$ gives $\mathfrak{b}(L_{2\mathscr{M}}) = \omega_1$. We see that these are relations (cardinals) involved in the so called *Cichoń's diagram.* Moreover, we have:

OBSERVATION 9. $\max\{\mathfrak{b}(L_{2\mathscr{M}}), \mathfrak{b}\} \le \min\{\mathrm{non}(\mathscr{K}), \mathfrak{d}\}$.

The idea of this paper is to present all results in a "durable" form, i.e., using Borel Galois-Tukey connections. To formulate inequalities as in the above observation in Galois-Tukey context we need relations representing max and min. Here an categorical motivation suffices (see [Vo3, Vo4]).

DEFINITION 10. Having two binary relations R and S we define

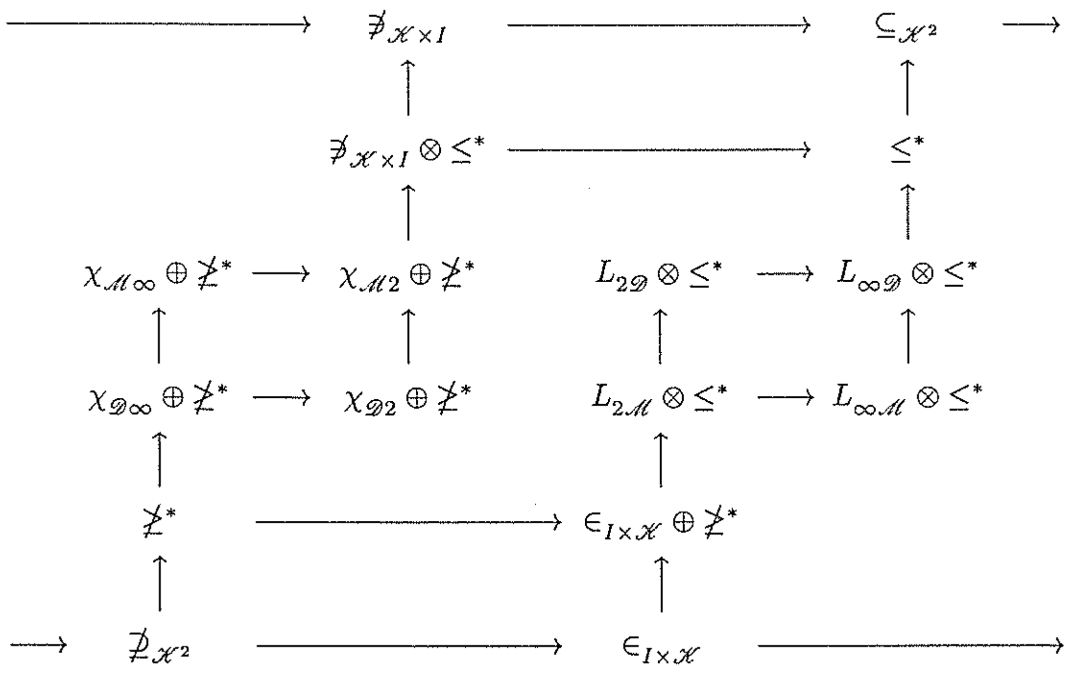
$$R \oplus S = \big\{\big((a,\varepsilon),(y,v)\big)\colon aRy \text{ and } \varepsilon = 0 \text{ or } aSv \text{ and } \varepsilon = 1\big\},$$
$$R \otimes S = \big\{\big((x,u),(a,\varepsilon)\big)\colon xRa \text{ and } \varepsilon = 0 \text{ or } uSa \text{ and } \varepsilon = 1\big\},$$
$$R \times S = \big\{\big((x,u),(y,v)\big)\colon xRy \text{ and } uSv\big\}.$$

We see that $\neg(R \otimes S)^{-1} = \neg R^{-1} \oplus \neg S^{-1}$, $\neg(R \oplus S)^{-1} = \neg R^{-1} \otimes \neg S^{-1}$ and

$$\mathfrak{b}(R \oplus S) = \min\big\{\mathfrak{b}(R), \mathfrak{b}(S)\big\} \quad \text{and} \quad \mathfrak{d}(R \oplus S) = \max\big\{\mathfrak{d}(R), \mathfrak{d}(S)\big\},$$
$$\mathfrak{b}(R \otimes S) = \max\big\{\mathfrak{b}(R), \mathfrak{b}(S)\big\} \quad \text{and} \quad \mathfrak{d}(R \otimes S) = \min\big\{\mathfrak{d}(R), \mathfrak{d}(S)\big\},$$

and $R \oplus S$ (with the natural embeddings and projections) is the categorical coproduct of R and S and $R \otimes S$ is the product (see [Vo4]).

Now we can widen our goal, not only to present inequalities between cardinal numbers in a "durable" way by Borel Galois-Tukey connections, but moreover to present all inequalities simultaneously by a diagram of Borel Galois-Tukey connections between relations. In [Vo3] we have done this for Cichoń's diagram. Here, so far we have done it for relations $\mathrm{LIM}_{*,*}$ and $\chi\mathrm{aos}_{*,*}$ situating them into Cichoń's diagram (see the diagram).



The "cardinal characteristic content" of the result of Kholshchevnikova and Malykhin was $\mathfrak{b}(L_{2\mathscr{M}}) \leq \mathfrak{d}$. We have improved this in [Vo1] by $\mathfrak{b}(L_{2\mathscr{M}}) \leq \mathfrak{b}\cdot\mathfrak{s}$ (note that $\mathfrak{b}\cdot\mathfrak{s} \leq \mathfrak{d}$ and consistently also $\mathfrak{b}\cdot\mathfrak{s} < \mathfrak{d}$, see [Va]). In [Vo3] we were able to represent our proof of $\mathfrak{b}(L_{2\mathscr{M}}) \leq \mathfrak{b}\cdot\mathfrak{s}$ by the so called max-min diagram, but we were not able to represent it by a Galois-Tukey connection. The problem whether inequalities proved by max-min diagram can be represented by Galois-Tukey connections was solved by A. Blass in [Bl1] and used in [Bl2]. Using Blass' result we can reformulate our result $\mathfrak{b}(L_{2\mathscr{M}}) \leq \mathfrak{b}\cdot\mathfrak{s}$ by a Galois-Tukey connection.

Definition 11 (A. Blass). For binary relations R and S we define the *sequential composition* $R;S$ of R and S (the order of R and S is important) as follows

$$\mathrm{dom}(R;S) = \mathrm{dom}(R) \times {}^{\mathrm{rng}(R)}\,\mathrm{dom}(S)\,,$$
$$\mathrm{rng}(R;S) = \mathrm{rng}(R) \times \mathrm{rng}(S)\,,$$

and for $x \in \mathrm{dom}(R)$, $f \in {}^{\mathrm{rng}(R)}\mathrm{dom}(S)$, $y \in \mathrm{rng}(R)$, $v \in \mathrm{rng}(S)$

$$\big((x,f),(y,v)\big) \in R;S \qquad \text{iff} \qquad (x,y) \in R \quad \text{and} \quad \big(f(y),v\big) \in S\,. \qquad (\Box)$$

Note that $\mathfrak{b}(R;S) = \min\{\mathfrak{b}(R),\mathfrak{b}(S)\}$, $\mathfrak{d}(R;S) = \max\{\mathfrak{d}(R),\mathfrak{d}(S)\}$ and (in the diagram the $\|$'s mean Galois-Tukey equivalence)

$$\begin{array}{ccccccc} R & \longrightarrow & R\oplus S & \longrightarrow & R\times S & \longrightarrow & R;S \\ & & \| & & \| & & \\ S & \longrightarrow & S\oplus R & \longrightarrow & S\times R & \longrightarrow & S;R \end{array}$$

THEOREM 12. *$\chi_{\mathscr{M}2} \to \not\geq^*;\chi_{\mathscr{D}2}$ and hence $\mathfrak{d}(\chi_{\mathscr{M}2}) \le \max\{\mathfrak{d}(\not\geq^*),\mathfrak{d}(\chi_{\mathscr{D},2})\}$ which is equivalent to $\mathfrak{b}(L_{2\mathscr{M}}) \le \mathfrak{b}\cdot\mathfrak{s}$.*

P r o o f. Instead of $\not\geq^*$ we consider $\neg\,\mathrm{EP}^{-1}$, which is Galois-Tukey equivalent to it. The E-mapping is facing a product and it has two components

$$E_1\colon \mathscr{M} \to [\omega]^\omega, \qquad E_2\colon \mathscr{M} \to {}^{\mathrm{Mon}}\mathscr{D}\,.$$

The F mapping we look for should fulfill $F\colon \mathrm{Mon}\times({}^\omega 2\setminus c) \to ({}^\omega 2\setminus c)$ in such a way that whenever $f \in \mathrm{Mon}$ is such that $\exists^\infty n\ \big|[f(n),f(n+1))\cap E_1(A)\big| \ge 2$, and $g \in {}^\omega 2$ does not have limit according to $E_2(A)(f) \in \mathscr{D}$, then $F(f,g)$ does not have limit according to matrix A. (Recall that the main point of this is the A. B l a s s' construction how to translate our max-min proof into Galois-Tukey fashion — the combinatorial essence remains the same as in [Vo1].) Namely $E_1(A) = \{\ell_n^A\colon n\in\omega\} \in [\omega]^\omega$ (as in [MKh] and also in [Vo1] and [Co]) and $E_2(A)(f) = \{n\colon \big|[f(n),f(n+1))\cap E_1(A)\big| \ge 2\} = M_f^A$ of [Vo1]. In fact $E_2(A)(f) \in [\omega]^\omega$, but we understand it in such a way, that the enumeration function $e_f^A \in \mathrm{Mon}$ of the set $E_2(A)(f)$ generates a diagonal matrix $D_{e_f^A}$. The fact that g does not have limit according to $D_{e_f^A}$ is equivalent to the fact that g restricted to $E_2(A)(f)$ does not have limit. $F(f,g)(k) = 0$ if $k \in [f(n),f(n+1))$ and $g(n) = 0$, else $F(f,g)(k) = 1$ (this is $g_{f,s}$ of [Vo1]). □

So using sequential composition of A. B l a s s and the combinatorial content of [Vo1] we were able to translate $\mathfrak{b}(L_{2\mathscr{M}}) \le \mathfrak{b}\cdot\mathfrak{s}$ to $\chi_{\mathscr{M}2} \to \not\geq^*;\chi_{\mathscr{D}2}$.

The situation becomes more interesting if we ask whether $\not\geq^*;\chi_{\mathscr{D}2}$ fits into Cichoń's diagram (as $\mathfrak{b}\cdot\mathfrak{s}$ does).

PROBLEM. *Is $\not\geq^*;\chi_{\mathscr{D}2}$ simpler (in a Borel way) than $\not\ni_{\mathscr{K}\times I}\otimes\le^*$?*

Let us note that there is another (a little bit cheating) possibility of doing so, just observe that $\chi_{\mathscr{M}2} \to \chi_{\mathscr{M}2}\oplus \not\geq^* \to \not\geq^*;\chi_{\mathscr{D}2}$ and hence the following relationship can be inserted for the corresponding edge of the Cichoń diagram.

$$\cdots\chi_{\mathscr{M}2}\oplus\not\geq^* \to (\not\geq^*;\chi_{\mathscr{D}2})\otimes(\not\ni_{\mathscr{K}\times I}\otimes\le^*) \to (\not\ni_{\mathscr{K}\times I}\otimes\le^*)\cdots$$

In terms of cardinal characteristics this means the following inequalities.

$$\mathfrak{b}(L_{2\mathscr{M}}) \leq \mathfrak{b} \cdot \mathfrak{b}(L_{2\mathscr{M}}) \leq \min\{\mathfrak{b} \cdot \mathfrak{s}, \min\{\mathrm{non}(\mathscr{K}), \mathfrak{d}\}\} = \mathfrak{b} \cdot \mathfrak{s} \leq \min\{\mathrm{non}(\mathscr{K}), \mathfrak{d}\}\,.$$

There is yet another aspect of our investigation. Knowing $\chi_{\mathscr{D}\infty} \to \chi_{\mathscr{D}2}$ (i.e., comparing the action of the subsequence limit method on different sets of sequences) one can ask: Is $\chi_{\mathscr{D}\infty}$ substantially simpler than $\chi_{\mathscr{D}2}$? Substantially simpler means that there is no Borel morphism from $\chi_{\mathscr{D}2}$ into $\chi_{\mathscr{D}\infty}$. So far we know $\mathfrak{d}(\chi_{\mathscr{D}2}) = \mathfrak{d}(\chi_{\mathscr{D}\infty}) = \mathfrak{s}$ but for $\mathfrak{r} = \mathfrak{b}(\chi_{\mathscr{D}2}) \leq \mathfrak{r}_\sigma = \mathfrak{b}(\chi_{\mathscr{D}\infty})$ we do not know whether there is a counterfact consisting of cardinal inequalities, or not, see problem of [Vo2] and Problem 273 of [Va].

Our results here is a sort of comparison of different methods of limits acting on 0–1 sequences. We know $\chi_{\mathscr{D}2} \to \chi_{\mathscr{M}2}$ and $\chi_{\mathscr{M}2} \to \not\geq^*;\chi_{\mathscr{D}2}$ is almost the opposite; just $\not\geq^*$ makes problems. We would like to ask whether there can be a Borel morphism from $\chi_{\mathscr{M}2} \to \chi_{\mathscr{D}2}$ or not. Because of

$$\begin{array}{ccccc}
\chi_{\mathscr{M}2} & \longrightarrow & \chi_{\mathscr{M}2} \oplus \not\geq^* & \longrightarrow & \not\geq^*;\chi_{\mathscr{D}2} \\
\uparrow & & \uparrow & & \uparrow \\
\chi_{\mathscr{D}2} & \longrightarrow & \chi_{\mathscr{D}2} \oplus \not\geq^* & \longrightarrow & \chi_{\mathscr{D}2} \times \not\geq^*
\end{array}$$

our result gives a little bit more insight. We know that if $\mathfrak{b} \leq \mathfrak{s}$, then $\mathfrak{b}(L_{2\mathscr{M}}) \leq \mathfrak{b}(L_{2\mathscr{D}})$ and if $\mathfrak{d} \geq \mathfrak{r}$, then $\mathfrak{d}(L_{2\mathscr{M}}) \geq \mathfrak{d}(L_{2\mathscr{D}})$, hence we can interpret it as follows: If a condition — not involving Toeplitz matrices — is fulfilled, then cardinal characteristics indicate that $\chi_{\mathscr{M}2}$ can be simpler than $\chi_{\mathscr{D}2}$. Rephrasing this for (Borel) Galois-Tukey connections we can ask: Is it possible that there is a Borel Galois-Tukey connection from $\not\geq^*;\chi_{\mathscr{D}2}$ to $\chi_{\mathscr{D}2}$? (It is a stronger way of asking than to ask for $\mathfrak{b} \leq \mathfrak{s}$ and $\mathfrak{d} \geq \mathfrak{r}$, which is consistent.) If yes, this is a condition not involving Toeplitz matrices and implying that $\chi_{\mathscr{D}2}$ is not substantially simpler than $\chi_{\mathscr{M}2}$ (or, vice versa, Toeplitz limit would be not substantially more complicated than taking limits of subsequences).

There is yet another way to achieve this, namely, if we were able to strengthen our result to $\chi_{\mathscr{M}2} \oplus \not\geq^* \to \chi_{\mathscr{D}2} \times \not\geq^*$ (a question of M. Repický), then it would suffice to have consistently $\chi_{\mathscr{D}2} \times \not\geq^* \to \chi_{\mathscr{D}2}$ in order to get $\chi_{\mathscr{M}2}$ to be simpler than $\chi_{\mathscr{D}2}$ due to a condition not involving matrices.

4. Added in proof

Let $\chi_{\omega 2} \subseteq \mathrm{Mon} \times {}^{\omega}2$ be defined as follows: $(f, g) \in \chi_{\omega 2}$ if $\exists^\infty n$, $[f(n), f(n+1)) \subseteq g^{-1}(0)$ and $\exists^\infty n\, [f(n), f(n+1)) \subseteq g^{-1}(1)$. In [KW] A. Kamburelis and B. Węglorz proved that $\mathfrak{d}(\chi_{\omega 2}) = \mathfrak{b} \cdot \mathfrak{s}$. The combinatorial content of their proof gives

$$\chi_{\mathscr{D}2} \oplus \not\geq^* \to \chi_{\omega 2} \to \not\geq^*; \chi_{\mathscr{D}2}.$$

Moreover, E_1 of Theorem 12 with the identity backwards forms a Borel connection $\chi_{\mathscr{M}2} \to \chi_{\omega 2}$. So a consistent Borel connection from $\chi_{\omega 2}$ to $\chi_{\mathscr{D}2}$ would be another example implying $\chi_{\mathscr{M}2} \to \chi_{\mathscr{D}2}$.

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Mathematical Institute
Slovak Academy of Sciences
Jesenná 5
SK–041 54 Košice
SLOVAKIA
E-mail: vojtas@kosice.upjs.sk