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A LATTICE OF BINARY RELATIONS WITH POLARITY

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Dedicated to Professor J. Jakubík on the occasion of his 70th birthday

ABSTRACT. The class of all binary relations ordered by (generalized) Galois–Tukey connections forms a lattice, moreover equipped with a naturally defined dual endomorphism (polarity). We study structural properties of this lattice and several types of sublattices (existence of smallest, largest elements, atoms, size of antichains, self-duality, (in)equalities of terms, etc.) and formulate some problems.

In the study of set-theoretical cardinal characteristics of real analysis it turned out that the unifying approach is to study cardinal characteristics of binary relations. Moreover, a constructive way of their comparison and/or estimation is the use of the concept of the (generalized) Galois–Tukey connections (see [V]). The notation is the usual one; that of lattice theory coincides with [G] and that of set theory with [J].

Assume R is a binary relation, $\mathrm{dom}(R)$ denotes the domain and $\mathrm{rng}(R)$ denotes the range of R, $(x,y) \in \neg R$ iff $x \in \mathrm{dom}(R)$ & $y \in \mathrm{rng}(R)$ & $(x,y) \notin R$, R^{-1} is the inverse of R. In what follows we assume binary relations are fulfilling $\mathrm{dom}(R) = \mathrm{dom}(\neg R)$ and $\mathrm{rng}(R) = \mathrm{rng}(\neg R)$. Note that in this case $\neg(\neg R^{-1})^{-1} = R$. Denote the class of all binary relations fulfilling our assumptions by $\mathcal{B}in$.

A set $B \subseteq \mathrm{dom}(R)$ is said to be R–unbounded, provided there is no single $y \in \mathrm{rng}(R)$ such that for all $b \in B$ is $(b,y) \in R$. Similarly, $D \subseteq \mathrm{rng}(R)$ is said to be an R–dominating set if for every $x \in \mathrm{dom}(R)$ there is $d \in D$ with $(x,d) \in R$.

Many cardinal characteristics studied in real-analysis are of the form $\mathfrak{b}(R)$— being the minimum of all possible cardinalities of R–unbounded sets and $\mathfrak{d}(R)$— being the minimum of all possible sizes of R–dominating sets (e.g., the minimal size of a nonmeasurable subset of unit interval is equal to

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$\mathfrak{b}(\in \cap([0.1] \times \mathcal{L}))$, i.e., the $\mathfrak{b}$-number of the relation of set-theoretic membership restricted to the product of unit interval and the ideal of sets of measure zero, see [V]).

Note that from our assumptions on binary relations it follows that $\mathfrak{b}(R)$ and $\mathfrak{d}(R)$ are always defined and ≥ 2. Moreover, note that $\mathfrak{b}(R) = \mathfrak{d}(\neg R^{-1})$ and $\mathfrak{d}(R) = \mathfrak{b}(\neg R^{-1})$.

The key concept we need to define our lattice is the following:

DEFINITION 1. ([V]) Assume R and S are binary relations. An ordered pair of functions (E, F) is called a (generalized) *Galois–Tukey connection* (also abbreviated as a GT-connection) from R to S if the following holds:

(a) $E\colon \operatorname{dom}(R) \to \operatorname{dom}(S)$,

(b) $F\colon \operatorname{rng}(S) \to \operatorname{rng}(R)$,

(c) $(\forall x \in \operatorname{dom}(R))$ $(\forall v \in \operatorname{rng}(S))$ $(E(x), v) \in S$ implies $(x, F(v)) \in R$.

We denote $R \preceq_{GT} S$ if there is a GT-connection from R to S.

Observe that $R \preceq_{GT} S$ implies $\mathfrak{b}(S) \leq \mathfrak{b}(R)$ and $\mathfrak{d}(S) \geq \mathfrak{d}(R)$. This is the main application of connections. Moreover, note that $R \preceq_{GT} S$ if and only if $\neg S^{-1} \preceq_{GT} \neg R^{-1}$, hence the mapping $R \to \neg R^{-1}$ is a self dual endomorphism (also called polarity) of the partially (pre)ordered class $\mathcal{B}in, \preceq_{GT}$.

We easily see that $\preceq_{GT}$ is reflexive and transitive on $\mathcal{B}in$. Of course it is not antisymmetric (e.g., the linear order of reals and the well-ordering of $\aleph_0$). Therefore, we have to mod-out relations under equivalence $R \approx_{GT} S$ defined to hold if both $R \preceq_{GT} S$ and $S \preceq_{GT} R$ hold. Note that the ordering of equivalence classes, namely $[R]_{\approx_{GT}} \leq_{GT} [S]_{\approx_{GT}}$ if $R \preceq_{GT} S$ is well defined. In the sequel we omit the notation $[R]_{\approx_{GT}}$ and instead of $\mathcal{B}in/_{\approx_{GT}}$ we will still write $\mathcal{B}in$ (omitting the way a representative of (a proper class) $[R]_{\approx_{GT}}$ is chosen).

To show that $\leq_{GT}$ is a lattice ordering we need the following

DEFINITION 2. ([V]) Assume R and S are binary relations, then we put

$$R \otimes S = \{((x, u), (a, \varepsilon)) : xRa \ \& \ \varepsilon = 0 \quad \text{or} \quad uSa \ \& \ \varepsilon = 1\},$$

and

$$R \oplus S = \{((a, \varepsilon), (y, v)) : aRy \ \& \ \varepsilon = 0 \quad \text{or} \quad aSv \ \& \ \varepsilon = 1\}.$$

THEOREM 1. *$R_1 \otimes R_2 = \inf_{\leq_{GT}}\{R_1, R_2\}$ and $R_1 \oplus R_2 = \sup_{\leq_{GT}}\{R_1, R_2\}$.*

P r o o f . Let us prove the first equality, the second is similar. Clearly, $R_1 \otimes R_2$ is $\leq_{GT}$-smaller than both R_1 and R_2 (just take the E-mapping being the corresponding projection of cartesian product of domains and the F-mapping the embedding of the appropriate range into the disjoint union of ranges).

Let T be a binary relation. which is $\leq_{GT}$-smaller than both R_1 and R_2, i.e., for $i = 1.2$ there are $H_i\colon \operatorname{dom}(T) \to \operatorname{dom}(R_i)$ and $K_i\colon \operatorname{rng}(R_i) \to \operatorname{rng}(T)$ with

$H_i(w)R_iy$ implying $wTK_i(y)$. Then the product of mappings (H_1, H_2) maps $\operatorname{dom}(T)$ into $\operatorname{dom}(R_1) \times \operatorname{dom}(R_2) = \operatorname{dom}(R_1 \otimes R_2)$ and the join of mappings $K_1 \cup K_2$ defined by glueing K_i's together on disjoint copies of ranges, i.e., on $\operatorname{rng}(R_1) \times \{0\} \cup \operatorname{rng}(R_2) \times \{1\} = \operatorname{rng}(R_1 \otimes R_2)$ fulfil necessary implication and therefore emphasizes $T \leq_{GT} R_1 \otimes R_2$. □

Thus by lattice theory (see [G]), $\leq_{GT}$ is a lattice-partial order, $\otimes$ is the meet and $\oplus$ is the join of corresponding algebraic lattice (on a proper class of all binary relations modulo $\approx_{GT}$).

Observe that in a similar way as in Theorem 1 (using the fact that our lattice consists of all relations) we can show that this lattice admits arbitrary joins and meets; of course the size of $\bigotimes \mathcal{R}$ and $\bigoplus \mathcal{R}$ can increase significantly (so sublattices restricted to sets are usually no more complete).

In applications in real analysis we work just with restricted part of our lattice.

Definition 3. Let Card be the class of all cardinal numbers and $X, Y \subseteq$ Card. Then put

$$\mathcal{B}in_X^Y = \{R \in \mathcal{B}in : \mathfrak{b}(R) \in X \text{ and } \mathfrak{d}(R) \in Y\}.$$

Let Card be seen as well-ordered class forms a (max-min) lattice, Card^{-1} is the dual lattice, then

$$\mathcal{C} = \mathrm{Card}^{-1} \times \mathrm{Card} \quad \text{is the direct product of lattices.}$$

To be dogmatically pure from the set theoretical point of view, let us note that in the sequel whenever we write, e.g., $(\mathcal{B}in, \otimes, \oplus)$ (or any n-tuple where entries are (possibly) proper classes) we understand this as an abbreviation, and in fact we are working with the formula of three variables (or n variables) defining this object.

Lemma.

(1) *The mapping $\mathcal{D}: \mathcal{B}in \to \mathcal{B}in$ defined by $\mathcal{D}(R) = \neg R^{-1}$ is (injective) dual endomorphism of the lattice $(\mathcal{B}in, \otimes, \oplus)$.*

(2) *$\mathcal{D}$ is a dual endomorphism of each $\mathcal{B}in_{\{\lambda\}}^{\{\lambda\}}$ and a dual homomorphism of $\mathcal{B}in_{\{\kappa\}}^{\{\lambda\}}$ onto $\mathcal{B}in_{\{\lambda\}}^{\{\kappa\}}$.*

Proof. Observe that $R \otimes S = \neg(\neg R^{-1} \oplus \neg S^{-1})^{-1}$ and $R \oplus S = \neg(\neg R^{-1} \otimes \neg S^{-1})^{-1}$ and recall that $\mathfrak{b}(R) = \mathfrak{d}(\neg R^{-1})$ and $\mathfrak{d}(R) = \mathfrak{b}(\neg R^{-1})$ holds. □

We show that estimating set-theoretical cardinal characteristics (though it can depend on models of set theory) forms a lattice homomorphism.

THEOREM 2. *The mapping $\mathcal{C}\colon \mathcal{B}in \to \mathcal{C}$ defined by $\mathcal{C}(R) = \big(\mathfrak{b}(R), \mathfrak{d}(R)\big)$ is a lattice homomorphism. Every $\mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$ is a sublattice of $\mathcal{B}in$.*

Proof. Notice that, e.g., $B \subseteq \mathrm{dom}(R \otimes S) = \mathrm{dom}(R) \times \mathrm{dom}(S)$ is $R \otimes S$–unbounded iff both projections B_x and B^u are R– and S–unbounded. So we have

$$\mathfrak{b}(R \otimes S) = \max\big(\mathfrak{b}(R), \mathfrak{b}(S)\big), \ \ \mathfrak{d}(R \otimes S) = \min\big(\mathfrak{d}(R), \mathfrak{d}(S)\big),$$

and

$$\mathfrak{b}(R \oplus S) = \min\big(\mathfrak{b}(R), \mathfrak{b}(S)\big), \ \ \mathfrak{d}(R \oplus S) = \max\big(\mathfrak{d}(R), \mathfrak{d}(S)\big).$$

□

This Theorem helps us to imagine how $\mathcal{B}in, \leq_{GT}$ looks like: namely like $\mathcal{C} = \mathrm{Card}^{-1} \times \mathrm{Card}$, where in place of (κ, λ) sits $\mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$. Moreover both are symmetric across the diagonal. We will see that for $(\kappa, \lambda) \leq_{\mathcal{C}} (\mu, \nu)$ the $\leq_{GT}$-ordering between $\mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$ and $\mathcal{B}in^{\{\nu\}}_{\{\mu\}}$ is neither total nor empty. Of course it does not say anything about the structure inside of $\mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$. In the sequel we try to get more information about the inner structure of $\mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$. Let $(=_{\mathrm{Real}})$ be the equality on reals, let $[2, \mathfrak{c}]$ be the interval of cardinal numbers κ with $2 \leq \kappa \leq \mathfrak{c} = 2^{\aleph_0}$ and $[X]^\kappa = \{Y \subseteq X \colon \ |\,Y\,| = \kappa\}$.

THEOREM 3.

(1) *Assume R is such that $\mathfrak{b}(R)$, $\mathfrak{d}(R) \leq 2^{\aleph_0}$. Then*

$$(\neq_{\mathrm{Real}}) \leq_{GT} R \leq_{GT} (=_{\mathrm{Real}}),$$

and so $\Big\langle \mathcal{B}in^{[2,\mathfrak{c}]}_{[2,\mathfrak{c}]}, \otimes, \oplus, (\neq_{\mathrm{Real}}), (=_{\mathrm{Real}})\Big\rangle$ is a lattice with the smallest and the largest element.

(2) *If $(=_{\mathrm{Real}}) \leq_{GT} R$, then there is an $X \in \big[\mathrm{dom}(R)\big]^{\geq \mathfrak{c}}$ such that $(R \restriction X)^{-1}$ is a function.*

Proof. (i) Fix a dominating family $D \subseteq \mathrm{rng}(R)$ of size $\leq 2^{\aleph_0}$ and a mapping F of Reals onto D. Define (by AC) $E(x) \in \big\{a\colon xRF(a)\big\}$. Then whenever we have $E(x) = a$ it implies $xRF(a)$, so (E, F) is a connection from R to $(=_{\mathrm{Real}})$. For the second GT–inequality note that $\mathfrak{d}(\neg R^{-1}) = \mathfrak{b}(R) \leq \mathfrak{c}$ gives $\neg R^{-1} \leq_{GT} (=_{\mathrm{Real}})$ and hence $(\neq_{\mathrm{Real}}) \leq_{GT} R$.

(ii) Let $E\colon \mathrm{Real} \to \mathrm{dom}(R)$ and $F\colon \mathrm{rng}(R) \to \mathrm{Real}$ forms a GT–connection. Then as every two-point set of reals is $=$-unbounded, the mapping E is one-to-one. We show that $\big(R \cap (\mathrm{rng}(E) \times R(\mathrm{rng}(E)))\big)^{-1}$ is a function. Assume not, i.e., there are $a \neq b$ reals and $y \in \mathrm{rng}(R)$ with $(E(a), y) \in R$ and $\big(E(b), y\big) \in R$, then by implication witnessing (E, F) is a connection we have $a = F(y) = b$, a contradiction. □

COROLLARY.

(1) *The lattice* $\left\langle \mathcal{B}in^{[2,\mathfrak{c}]}_{[2,\mathfrak{c}]}, \otimes, \oplus, (\neq_{\text{Real}}), (=_{\text{Real}}) \right\rangle$ *as a structure of the language* $(L, \wedge, \vee, 0, 1)$ *does not admit complementation and even fulfills*
$(\forall x)(\forall y)(x > 0 \ \& \ y > 0 \implies x \wedge y > 0)$ *and*
$(\forall x)(\forall y)(x < 1 \ \& \ y < 1 \implies x \vee y < 1)$,

(2) $\mathcal{B}in^{\{2\}}_{\{\mathfrak{c}\}}$ *(and hence* $\mathcal{B}in^{[2,\mathfrak{c}]}_{[2,\mathfrak{c}]}$ *too) has at most one atom,*

(3) $\mathcal{B}in, \leq_{GT}$ *has neither smallest nor largest element.*

Proof. (1) Assume R and S are $\leq_{GT}$-strictly smaller than $1 = (=_{\text{Real}})$. Notice (roughly speaking) that if neither R^{-1} nor S^{-1} contains a function with range of size at least $\mathfrak{c}$, then $(R \oplus S)^{-1}$ does not contain such a function either, simply because $R \oplus S$ is like a disjoint union of cylinders $R \times \text{rng}(S)$ and $S \times \text{rng}(R)$ and its domain is the disjoint union of domains of R and S. The dual assertion follows after applying the dual endomorphism $\mathcal{D}$.

(2) Straightforward using (1).

(3) For every candidate R take $\kappa = |R|^+$, then $(=_\kappa)$ and/or $(\neq_\kappa)$ is a counterexample. □

THEOREM 4. *Assume* κ, λ *are infinite regular cardinal numbers. Then* $\mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$ *contains an antichain which is a proper class.*

Proof. Take $\kappa, \lambda, \mu, \nu$ infinite regular cardinals such that $\kappa < \lambda < \mu < \nu$ and $\leq_\kappa$, $\leq_\lambda$, $\leq_\mu$, $\leq_\nu$ are canonical well-orderings of this cardinals. Put

$$R_1 = (\leq_\kappa \otimes \leq_\lambda) \oplus (\leq_\lambda \otimes \leq_\mu),$$
$$S_1 = (\leq_\kappa \otimes \leq_\lambda) \oplus (\leq_\lambda \otimes \leq_\nu),$$
$$R_2 = (\leq_\kappa \otimes \leq_\lambda) \oplus (\leq_\kappa \otimes \leq_\mu),$$
$$S_2 = (\leq_\kappa \otimes \leq_\lambda) \oplus (\leq_\kappa \otimes \leq_\nu).$$

We show that

(1) $R_1, S_1 \in \mathcal{B}in^{\{\lambda\}}_{\{\lambda\}}$ and $R_1 \not\leq_{GT} S_1$ and $S_1 \not\leq_{GT} R_1$ and

(2) $R_2, S_2 \in \mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$ and $R_2 \not\leq_{GT} S_2$ and $S_2 \not\leq_{GT} R_2$.

Computations of $\mathfrak{b}(R_i)$, $\mathfrak{d}(R_i)$ are straightforward by Theorem 2.

(1) We show that $R_1 \not\leq_{GT} S_1$. Assume, by contradiction, yes, i.e., $R_1 \leq_{GT} S_1$. Then as R_1 is a join of two relations, this is equivalent to both

$$(\leq_\kappa \otimes \leq_\lambda) \leq_{GT} (\leq_\kappa \otimes \leq_\lambda) \oplus (\leq_\lambda \otimes \leq_\nu),$$

(which is always true as $a \leq a \vee b$ holds in every lattice) and

$$(\leq_\lambda \otimes \leq_\mu) \leq_{GT} (\leq_\kappa \otimes \leq_\lambda) \oplus (\leq_\lambda \otimes \leq_\nu).$$

So far as in every lattice holds $(b \wedge a) \vee (a \wedge c) \leq a \wedge (b \vee c)$ (and we apply this to the right-hand side of the last inequality), this implies

$$(\leq_\lambda \otimes \leq_\mu) \leq_{GT} (\leq_\lambda) \otimes (\leq_\kappa \oplus \leq_\nu).$$

As the right side of this is a meet of two relations, this is equivalent to both

$$(\leq_\lambda \otimes \leq_\mu) \leq_{GT} (\leq_\lambda),$$

(which is always true by $a \wedge b \leq a$) and

$$(\leq_\lambda \otimes \leq_\mu) \leq_{GT} (\leq_\kappa \oplus \leq_\nu),$$

which should lead to contradiction.

So we assume there are two mappings

$$E\colon \lambda \times \mu \to \kappa \times \{0\} \cup \nu \times \{1\} \quad \text{and}$$
$$F\colon \kappa \times \nu \to \lambda \times \{0\} \cup \mu \times \{1\}$$

fulfilling the implication

$$E(\alpha,\beta) \bigoplus (\gamma,\delta) \implies (\alpha,\beta) \bigotimes F(\gamma,\delta).$$

Here $\bigoplus$ and $\bigotimes$ just schematically abbreviate corresponding relations. As $|\lambda \times \mu| = \mu < \left|\kappa \times \{0\} \cup \nu \times \{1\}\right| = \nu$ and ν is a regular cardinal number, the set $E(\lambda \times \mu) \cap (\nu \times \{1\})$ is $\bigoplus$–bounded by $(7, \alpha_0)$ for some $\alpha_0 \in \nu$ (this $\bigoplus$-boundedness does not depend on the first coordinate of $(7, \alpha_0)$ because the very set on the left has second coordinate equal to 1, which is pointing to the second coordinate, namely α_0, on the right. The number 7 we just have chosen to stress this). So

$$E^{-1}(\nu \times \{1\}) = \lambda \times \mu \backslash E^{-1}(\kappa \times \{0\}) \bigotimes F(7, \alpha_0),$$

($F(7,\alpha_0)$ is a bound in the λ or μ coordinate). So there are either $\alpha_1 \in \lambda$ or $\beta_1 \in \mu$ such that either $E\big((\alpha_1, \lambda) \times \mu\big) \subseteq \kappa \times \{0\}$ or $E\big(\lambda \times (\beta_1, \mu)\big) \subseteq \kappa \times \{0\}$. But in both cases there is $Y_0 \in [\mu]^\mu$ such that for every $\beta \in Y_0$ there is an $X_\beta \in [\lambda]^\lambda$ and $\gamma_\beta \in \kappa$ such that $E\big(X_\beta \times \{\beta\}\big) = \gamma_\beta$. As $\mu > \kappa$ there is a $Y_1 \in [Y_0]^\mu$ such that for $\beta_1, \beta_2 \in Y$ is $\gamma_{\beta_1} = \gamma_{\beta_2} = \gamma^*$. But then

$$E\Big(\bigcup_{\beta \in Y_1} X_\beta \times \{\beta\}\Big) = (\gamma^*, 0) \bigoplus (\gamma^* + 1, 7),$$

a contradiction because $\bigcup_{\beta \in Y_1} X_\beta \times \{\beta\}$ is unbounded in both coordinates and simultaneously $\bigotimes$–bounded by $F(\gamma^* + 1, 7)$.

Now, we show $S_1 \not\leq_{GT} R_1$. Similarly as in the first case, assume by contradiction that it is so, and by similar reductions this leads to

$$(\leq_\lambda \otimes \leq_\nu) \leq_{GT} (\leq_\kappa \oplus \leq_\mu),$$

which should give a contradiction.

Now the situation is similar (though not identical) with the previous case. We try to be brief and still precise. So we assume there are

$$E\colon \lambda \times \nu \to \kappa \times \{0\} \cup \mu \times \{1\} \quad \text{and}$$
$$F\colon \kappa \times \mu \to \lambda \times \{0\} \cup \nu \times \{1\},$$

fulfilling appropriate implication. Take $Y \subseteq \lambda$ such that for every $\alpha \in Y$ there is an $X_\alpha \in [\nu]^\nu$ and $\beta_\alpha \in \mu$ with $E(\{\alpha\} \times X_\alpha) = (\beta_\alpha, 1)$. As $\lambda < \mu$ there is $\beta^* \in \mu$ bounding all $\{\beta_\alpha \colon \alpha \in \lambda\}$. So

$$E\Big(\bigcup_{\alpha\in Y} \{\alpha\} \times X_\alpha\Big) \bigoplus (7, \beta^*) \quad \text{and} \quad \Big(\bigcup_{\alpha\in Y} \{\alpha\} \times X_\alpha\Big) \bigoplus F(7, \beta^*).$$

Hence, as X_α's are unbounded, Y is bounded. So there is an $\alpha_0 \in \lambda$ such that for all $\alpha \in (\alpha_0, \lambda)$ and for all $\beta \in \mu$ is $\left|E^{-1}(\beta, 1) \cap (\{\alpha\} \times \nu)\right| < \nu$ and as $\mu < \nu$ also $\left|E^{-1}(\mu \times \{1\}) \cap (\{\alpha\} \times \nu)\right| < \nu$. So there are $\delta_\alpha \in \kappa$ and $Z_\alpha \in [\nu]^\nu$ such that $E\big(\{\alpha\} \times Z_\alpha\big) = (\delta_\alpha, 0)$. This leads to a contradiction in an identical way as in the first case. Note that to get the result for $\mathcal{B}in^{\{\aleph_0\}}_{\{\aleph_0\}}$ we can take κ to be finite.

(2) As in the previous steps assumption $R_2 \leq_{GT} S_2$ leads to assumption

$$(\leq_\kappa \otimes \leq_\mu) \leq_{GT} (\leq_\lambda \oplus \leq_\nu),$$

and assumption $S_2 \leq_{GT} R_2$ leads to assumption

$$(\leq_\kappa \otimes \leq_\nu) \leq_{GT} (\leq_\lambda \oplus \leq_\mu).$$

Both give a contradiction in a similar way as in previous two cases. Thus we get the result for $\mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$ with $\kappa < \lambda$. For $\kappa > \lambda$ just use the dual isomorphism $\mathcal{D}$. □

COROLLARY. *For every $(\kappa, \lambda) \leq_C (\mu, \nu)$ there are $R \in \mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$ and $S \in \mathcal{B}in^{\{\nu\}}_{\{\mu\}}$ such that $R \not\leq_{GT} S$.*

Proof. In the previous theorem we showed, e.g.,

$$(\leq_\lambda \otimes \leq_\mu) \not\leq_{GT} (\leq_\kappa \oplus \leq_\nu),$$

(for $(\lambda, \mu) \leq_C (\kappa, \nu)$ in the notation of Theorem 4). In a similar way discussing all possible positions between $\kappa, \lambda, \mu, \nu$ we get the result. □

Observe that whenever $(\kappa,\lambda) \le_C (\mu,\nu)$ then for every $R \in \mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$ ($\mathcal{B}in^{\{\nu\}}_{\{\mu\}}$, respectively) there is an $S \in \mathcal{B}in^{\{\nu\}}_{\{\mu\}}$ ($\mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$, resp.) such that $R \le_{GT} S$ ($\ge_{GT}$, resp.). That is, the lattice $\mathcal{B}in$ looks like C with $\mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$ in the node (κ,λ), the ordering between $\mathcal{B}in^{\{\lambda\}}_{\{\kappa\}}$ and $\mathcal{B}in^{\{\nu\}}_{\{\mu\}}$ being neither total nor empty (if (κ,λ) and (μ,ν) are C–comparable).

OBSERVATION. *Our example* $\langle \mathcal{B}in, \otimes, \oplus, \mathcal{D}, \ne_{\mathrm{Real}}, =_{\mathrm{Real}} \rangle$ *and all substructures restricted to some* $\mathcal{B}in^X_X$ *are structures of the language* $(L, \wedge, \vee, {}^o, 0, 1)$, ^{o} *being a unary operation, and fulfill all conditions necessary to be de Morgan, Stone and/or MS–algebras (see* [Va]) *but distributivity.*

CONJECTURE. $(\mathcal{B}in, \le_{GT})$ *is not distributive.*

PROBLEM. Decide (describe) further structural properties of complete class-lattices $\mathcal{B}in^Y_X$ (e.g., atoms, existence of smallest, largest element, isomorphism of them, which equalities and/or inequalities hold (does not hold, respectively)).

Study this questions also for $\mathcal{B}in_{\mathrm{Real}} = \{R\colon \operatorname{dom}(R) \cup \operatorname{rng}(R) \subseteq \mathrm{Real}\}$ or for $\mathcal{B}in_{\mathrm{Borel}} = \{R\colon R \subseteq \mathrm{Borel}(\mathrm{Real} \times \mathrm{Real})\}$ (Defining the lattice operations by taking reasonable representatives of appropriate equivalence classes). These are lattices with the underlying class being set, all results of this paper but Corollary (3) of Theorem 3 and Theorem 4 with Corollary are valid (though certain portion of these is also valid) also for these set-supported lattices (which are no more complete).

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