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GENERALIZED GALOIS-TUKEY-CONNECTIONS BETWEEN EXPLICIT RELATIONS ON CLASSICAL OBJECTS OF REAL ANALYSIS

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Abstract. In this survey we are concerned with the study of structures and properties of classical real analysis as Lebesgue measure, Baire category, limit of sequences and absolute convergence of series and we introduce a new (unified) treatment of phenomena which covers most of results in the field. To compare additivity of measure and category, D.H.Fremlin first used connections in this context. We generalize his approach to the study of connections between relations (inclusion, membership, existence of limit, absolute summability) restricted to classical objects of real analysis (ideals of null and meagre sets, reals, ℓ^1, ℓ^∞ and $\wp(\omega)$). We discuss also set-theoretic absoluteness of some connections in the context of properties of forcing extensions and aspects of the algebraic theory of categories.

1. Introduction.

The study of cardinal characteristics of structures connected with the real line is an important part of applications of set theory to real analysis. The update survey on these cardinals is in J.E.Vaughan's article "Small uncountable cardinals in topology" ([Va]). Most of these numbers are laying between $\aleph_1$ and $2^{\aleph_0}$ and so interrelations between them are not interesting under CH. We present an approach which is interesting also under CH.

Let $\mathbb{L}$ denotes the ideal of Lebesgue measure zero sets and $\mathbb{K}$ the ideal of sets of first Baire category. In his lecture for Séminaire d'Initiation á l'Analyse at Paris 6 with title "Cichoń's diagram" ([Fr1]) D.H.Fremlin (depending on ideas of J.Cichoń and J.Pawlikowski and on [Ba] and [RaSt]) constructed a pair of mappings $E : \mathbb{K} \longrightarrow \mathbb{L}$ and $F : \mathbb{L} \longrightarrow \mathbb{K}$ such that $E(A) \subseteq B$ implies $A \subseteq F(B)$.

As a corollary he obtained the following observation: If $\mathfrak{A} \subseteq \mathbb{K}$ is such that $\bigcup \mathfrak{A} \notin \mathbb{K}$ then $\bigcup E(\mathfrak{A}) \notin \mathbb{L}$. So the additivity of Lebesgue measure is less than or equal the additivity of category (originaly proved by T.Bartoszynski ([Ba]) and J.Raisonier and J.Stern ([RaSt])). The existence of these mappings is interesting

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also under CH and in special cases coincides with the notion of Galois connections (see [HeHu]) and Tukey functions ([Tu], see [Fr2]). That is why we call our notion generalized Galois-Tukey connections.

In this introduction we try to motivate the general setting in which we consider cardinal characteristics of relations and generalized Galois-Tukey connections between relations by the following example. First some terminology. A family $\mathfrak{A} \subseteq \mathbb{K}$ such that $\bigcup \mathfrak{A} \notin \mathbb{K}$ is unbounded in the partially ordered set $(\mathbb{K}, \subseteq)$, that means, there is no $B \in \mathbb{K}$ such that for every $A \in \mathfrak{A}$, $A \subseteq B$ holds. Motivated by the following result of F.Rothberger ([Ro1]) we generalize the concept of an unbounded subset of a partially ordered set to the concept of an R-unbounded set for a binary relation R. A set $B \subseteq \mathrm{dom}(R)$ is said to be R-unbounded if $(\forall y \in \mathrm{rng}(R))(\exists x \in B)((x, y) \notin R)$. A set $D \subseteq \mathrm{rng}(R)$ is said to be R-dominating if $(\forall x \in \mathrm{dom}(R))(\exists y \in D)((x, y) \in R)$. As $\mathrm{cov}(\mathbb{L})$ is the minimum of all possible sizes of families $\mathfrak{A} \subseteq \mathbb{L}$ such that $\bigcup \mathfrak{A} = \mathcal{R}$ and $\mathrm{non}(\mathbb{K})$ is the minimum of all possible sizes of $X \subseteq \mathcal{R}$ such that $X \notin \mathbb{K}$, so observe that $\mathrm{non}(\mathbb{K})$ is the minimal size of an $\in$-unbounded family, where the relation $\in$ (ordinary set-theoretic membership) is restricted to $[0, 1] \times \mathbb{K}$ and $\mathrm{cov}(\mathbb{L})$ is the minimal size of $\in$-dominating family where the relation $\in$ is restricted to $[0, 1] \times \mathbb{L}$. (Please notice the orientation of relations, R-unbounded sets are in the first coordinate (domain) and R-dominating in the second (range)). F.Rothberger proved in ([Ro1]) that $\mathrm{cov}(\mathbb{L}) \leq \mathrm{non}(\mathbb{K})$. Moreover the proof of this inequality can be carried out (similarly as in Fremlin's proof) by a (generalized) connection. Just take $A \in \mathbb{L}$ such that A is comeager and let $B = \mathcal{R} \setminus A$. Let $E' : \mathcal{R} \longrightarrow \mathbb{L}$ and $F' : \mathcal{R} \longrightarrow \mathbb{K}$ be defined by putting $E'(x) = x + A$ and $F'(y) = y - B$ (where e.g. $x + A = \{x + y : y \in A\}$) and note that $x + A \not\ni y$ implies $x \in y - B$. In fact this is a connection between $\in$ and $\not\ni$, and $\mathrm{cov}(\mathbb{L})$ is also the minimal size of an $\not\ni$-unbounded family ($\not\ni$ restricted to $\mathbb{L} \times [0, 1]$).

In this survey paper we use the concept of generalized Galois-Tukey connection (sometimes abbreviated as GT-connection) for the study of relations involved in Cichoń's diagram and some other relations of real analysis such as existence of limit and absolute convergence of series. J.Steprans informed us that similar approach (to use GT-connections in generality) was considered also by A.W.Miller ([Mi3]), another unified treatment of the subject is in J.Pawlikowski's paper [Pa]. In §2 we give the definition of R-boundedness and R-dominatedness, claim basic properties, list all relations concerned and their interrelations to the cardinal characteristics of ([Va]). In §3 we give the definition of generalized Galois-Tukey connection between relations and some basic properties. We list known connections from literature. In §4 we discuss how inequalities between maximum and minimum of cardinal invariants can be carried out by GT-connections and we apply this to certain relations. In §5 we consider the possibility of set-theoretic absoluteness of mappings involved in the concept of GT-connections and apply this to the problem of the existence of R-unbounded (R-dominating) objects (e.g. reals) for some known forcing extensions. In §6 we discuss some aspects of our approach and give some open problems. Our notation is the standard set-theoretic one and we follow that of ([Je]) for set theory and partial orders, ([Va]) for cardinal characteristics and ([Fi]) for real analysis.

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2. Cardinal characteristics of binary relations.

2.1. The concept of R-unbounded and R-dominating sets, $\mathfrak{b}(R)$ and $\mathfrak{d}(R)$.

Let (x, y) denotes the ordered pair of x and y and the product $X \times Y = \{(x, y) : x \in X and\ y \in Y\}$, X^2 stands for $X \times X$. A binary relation R is a set (or also a proper class) of ordered pairs. The domain $\mathrm{dom}(R) = \{x : (\exists y)((x, y) \in R)\}$ and the range $\mathrm{rng}(R) = \{y : (\exists x)((x, y) \in R)\}$. The fact $(x, y) \in R$ will be often written as xRy. The complement (or negation) of R is denoted by

$$\neg R = \{(x, y) : x \in \mathrm{dom}(R) and\ y \in \mathrm{rng}(R) and\ (x, y) \notin R\}.$$

The inverse $R^{-1} = \{(y, x) : (x, y) \in R\}$. We emphasize that in what follows it will be important to notice the orientation of coordinates in relations and we try to introduce a "user friendly" notation.

2.1.1. Definitions. A set $B \subseteq \mathrm{dom}(R)$ is said to be <u>R-unbounded</u> if $(\forall y \in \mathrm{rng}(R))$ $(\exists x \in B)$ $((x, y) \notin R)$. A set $D \subseteq \mathrm{rng}(R)$ is said to be <u>R-dominating</u> if $(\forall x \in \mathrm{dom}(R))$ $(\exists y \in D)$ $((x, y) \in R)$. The corresponding "unboundedness"-like and "dominatedness" -like (abbreviated $\mathfrak{b}$ and $\mathfrak{d}$-like) cardinal characteristics are the following:

$$\mathfrak{b}(R) = \min\{|B| : B \subseteq \mathrm{dom}(R) and\ B \text{ is an } R\text{-unbounded set}\}$$

and

$$\mathfrak{d}(R) = \min\{|D| : D \subseteq \mathrm{rng}(R) and\ D \text{ is an } R\text{-dominating set }\}$$

We emphasize that unboundedness is involved in the first coordinate and dominatedness in second coordinate of R.

2.1.2. In order to avoid undefined or trivial cases we consider in the future only relations R such that $\mathrm{rng}(\neg R) = \mathrm{rng}(R)$ and $\mathrm{dom}(\neg R) = \mathrm{dom}(R)$ (when necessary we restrict R to some $X \times Y$). (See notes in 2.2.1., 3.2.3.1. and 5.1.2.) Note that in this case both $\mathfrak{b}(R)$, $\mathfrak{d}(R)$, $\mathfrak{b}(\neg R)$ and $\mathfrak{d}(\neg R)$ are defined and all are greater than or equal to 2 (see also next observation).

2.1.3. Some basic properties. As $(x, y) \notin R$ means also $(y, x) \in (\neg R^{-1})$ and $(x, y) \in R$ also $((y, x) \notin \neg R^{-1})$, there are only 4 numbers related to the relations R, $\neg R^{-1}$, $\neg R$ and $R^{-1} = \neg(\neg R)^{-1}$, and so the following is easy

OBSERVATION.

(1) $\mathfrak{b}(R) = \mathfrak{d}(\neg R^{-1})$
(2) $\mathfrak{d}(R) = \mathfrak{b}(\neg R^{-1})$
(3) $\mathfrak{b}(\neg R) = \mathfrak{d}(R^{-1})$
(4) $\mathfrak{d}(\neg R) = \mathfrak{b}(R^{-1})$

This is useful, first because we saw in Rothberger's result in the introduction that we needed both $\in$ and $\not\ni = \neg \in^{-1}$ and, second because some cardinal invariants were historically defined by some people in positive way (as for R) and by others in the negative way (as for $\neg R^{-1}$) (e.g. relations of "refining" ([Vo1]) and of "reaping" ([BevD]) leading to numbers $\mathfrak{r}$ and $\mathfrak{s}$; independently defined also by B.Balcar, J.Cichoń and J.van Mill, see [Va] and [St]).

2.2. EXAMPLES OF VARIOUS RELATIONS RESTRICTED TO DIFFERENT DOMAINS.

2.2.1. Definitions. Take a product $X \times Y$ and consider a binary relation R. A restriction of R to $X \times Y$ is the relation

$$R \cap (X \times Y) = \{(x,y) \in R : x \in X and\ y \in Y\}.$$

This restriction will be sometimes abbreviated as $R_{X\times Y}$, similarly $R \cap (X)^2$ and $R_{(X)^2}$. We now list definitions of some relations R and possible products $X \times Y$ to which R can be restricted and/or set-theoretic representation of objects of real analysis. For a real number x, $|x|$ denotes the absolute value of x, whereas for an arbitrary set X, $|X|$ is the cardinality of X. We hope that there will be no confusion.

SETS.

Basic sets.

$2 = \{0,1\}$ i.e. we consider the number 2 as the two element set consisting of 0 and 1;

ω is the set of all natural numbers;

$\mathcal{Q}$ is the set of all rational numbers;

$\mathcal{R}$ is the set of all real numbers,

$\mathcal{R}_+ = \{x \in \mathcal{R} : 0 \neq x\}$;

$\mathcal{I} = [0,1]$ is the unit interval which often represents reals when considering properties of measure and category.

Sequences.

${}^\omega X$ is the set of all mappings from ω into X considered as sequences (always countably infinite) of elements of X, in particular

${}^\omega 2$ is the set of all $0-1$ sequences, sometimes identified with $\mathcal{I} = [0,1]$ via dyadic expansions;

${}^\omega\omega$ is the set of all sequences of natural numbers which represents $[0,1]$ via identifying irationals with continuous fractions;

${}^\omega\mathcal{R}$ is the set of all sequences of real numbers, analogously ${}^\omega\mathcal{R}_+$ of nonzero entries.

Sets of subsets.

$[X]^\omega$ is the set of all countably infinite subsets of X, in particular

$[\omega]^\omega$ is the set of all infinite sets of natural numbers, sometimes also identified with the class of diagonal regular matrices (see bellow and the introduction to 3.2.2.).

Spaces.

$\ell^1 = \left\{a \in {}^\omega\mathcal{R} : \sum_{n=0}^{\infty} |a(n)| < +\infty\right\}$ is the set of all absolutely convergent series, similarly

$\ell^1_+ = \left\{a \in {}^\omega\mathcal{R}_+ : \sum_{n=0}^{\infty} |a(n)| < +\infty\right\}$;

$\ell^\infty = \{a \in {}^\omega\mathcal{R} : \limsup_{n\to+\infty} |a(n)| < +\infty\}$;

$c_0 = \{a \in \ell^\infty : \lim_{n\to+\infty} a(n) = 0\}$;

$h_0 = \{a \in \ell^\infty : \liminf_{n\to+\infty} |a(n)| = 0\}$.

Ideals.

$\mathbb{L} = \{X : X \subseteq [0,1] and\ \mu(X) = 0\}$;

$\mathbb{K} = \{X : X \subseteq [0,1] and\ X$ is of first category $\}$;

$\mathfrak{E} = \{X : X \subseteq [0,1] and\ X$ is closed set of measure zero $\}$;

$\mathfrak{F} = \{X : X \subseteq [0,1] and\ X$ is nowhere dense $\}$;

$\mathfrak{Z} = \{A : A \subseteq \omega and\ A$ is of asymptotic density zero $\}$.

Matrices.

$\mathbb{M} = \{C : C$ is a regular (Toeplitz) matrix $\}$ where for an infinite matrix $C = \{c(n,k) : n,k \in \omega\}$ of real numbers to be called regular (or also Toeplitz) is necessary and sufficient to fulfill the following conditions:

(1) $\exists m\, \exists l\, \forall n \geq l \quad \sum_{k=0}^{\infty} |c(n,k)| < m$

(2) $\forall k \lim_{n\to+\infty} c(n,k) = 0$

(3) $\lim_{n\to+\infty} \sum_{k=0}^{\infty} c(n,k) = 1.$

RELATIONS.

We now list some binary relations (some of them are proper classes) we investigate the existence of connections between their restrictions (which will be always sets).

Set-theoretic relations.

$\in = \{(x,y) : x \in y\}$ and $\not\ni = \{(x,y) : y \notin x\}$;

$\subseteq = \{(x,y) : x \subseteq y\}$ and $\not\supseteq = \{(x,y) : y \not\subseteq x\}$;

$\leq = \{(f,g) : f,g \in {}^\omega\mathcal{R} and\ \forall n |f(n)| \leq |g(n)|\}$.

Eventual relations.

$\supseteq^* = \{(x,y) : |y \setminus x| < \aleph_0\}$;

$\leq^* = \{(f,g) : f,g \in {}^\omega\mathcal{R} and\ \exists k \forall n \geq k |f(n)| \leq |g(n)|\}$;

$\not\geq^* = \{(f,g) : f,g \in {}^\omega\mathcal{R} and\ \forall k \exists n \geq k |f(n)| < |g(n)|\}$;

$\precsim^* = \left\{(a,b) : a,b \in \ell^1_+ and\ \exists k \forall n \geq k \left|\frac{a(n+1)}{a(n)}\right| \leq \left|\frac{b(n+1)}{b(n)}\right|\right\}$.

Existence of limits.
LIM $= \{(a,X) : a \in \ell^\infty and\ X \in [\omega]^\omega and\ \lim_{n\in X} a(n)$ exists $\}$;
χAOS $= \{(X,a) : a \in \ell^\infty and\ X \in [\omega]^\omega and\ \lim_{n\in X} a(n)$ does not exist$\}$;
For $a \in \ell^\infty$ and an infinite matrix C define

$$\operatorname{Clim}_{n\to+\infty} a(n) = \lim_{n\to+\infty} \sum_{k=0}^{\infty} c(n,k)a(k) \text{ if this limit exists .}$$

Recall (see[Co]) that the necessary and sufficient condition for the Clim to prolongate the usual lim is that C is regular.
RLIM $= \{(a,C) : a \in \ell^\infty and\ C \in \mathbb{M} and\ \operatorname{Clim}_{n\to+\infty} a(n)$ exists$\}$;
RχAOS $= \{(C,a) : a \in \ell^\infty and\ C \in \mathbb{M} and\ \operatorname{Clim}_{n\to+\infty} a(n)$ does not exist$\}$.
Series.
CONV $= \{(a,X) : a \in \ell^\infty and\ X \in [\omega]^\omega and\ \sum_{n\in X} |a(n)| < +\infty)\}$.
And so for instance

$$\not\ni \cap(\mathbb{L} \times [0,1]) = \{(X,x) : X \in \mathbb{L} and\ x \in \mathcal{I} and\ x \notin X\}$$

which we abbreviate as $\not\ni_{\mathbb{L}\times\mathcal{I}}$ or e.g.

$$\text{LIM} \cap ({}^\omega 2 \times [\omega]^\omega) = \{(a,X) \in \text{LIM} : a \in {}^\omega 2 and\ X \in [\omega]^\omega\}.$$

Notice. Please notice that our relations and sets to which their can be restricted all fulfill conditions of 2.1.2. except of CONV, which can be restricted only to sets which are disjoint with ℓ^1 (e.g. $c_0 \setminus \ell^1$, $h_0 \setminus \ell^1$) and all LIM-type relations which can be restricted to sets which are disjoint with the set of all convergent sequences in ℓ^∞ (especially set of all characteristic functions of finite or cofinite sets in ${}^\omega 2$, or cofinite sets in $[\omega]^\omega$). We do not complicate the notation, so when necessary we restrict ourself to smaller field to fulfill 2.1.2. (namely in 3.2.3.1. and 5.1.2.).

2.2.2. $\mathfrak{b}(R)$ and $\mathfrak{d}(R)$ related to the notation of van Douwen and Vaughan. The following holds:

THEOREM.

$$\begin{aligned}
\mathfrak{b} &= \mathfrak{b}(\le^* \cap({}^\omega\omega)^2), & \mathfrak{d} &= \mathfrak{d}(\le^* \cap({}^\omega\omega)^2);\\
\text{add}(\mathbb{L}) &= \mathfrak{b}(\subseteq \cap(\mathbb{L})^2), & \text{cof}(\mathbb{L}) &= \mathfrak{d}(\subseteq \cap(\mathbb{L})^2);\\
\text{add}(\mathbb{K}) &= \mathfrak{b}(\subseteq \cap(\mathbb{K})^2), & \text{cof}(\mathbb{K}) &= \mathfrak{d}(\subseteq \cap(\mathbb{K})^2);\\
\text{non}(\mathbb{L}) &= \mathfrak{b}(\in \cap([0,1]\times\mathbb{L})), & \text{cov}(\mathbb{L}) &= \mathfrak{d}(\in \cap([0,1]\times\mathbb{L}));\\
\text{non}(\mathbb{K}) &= \mathfrak{b}(\in \cap([0,1]\times\mathbb{K})), & \text{cov}(\mathbb{K}) &= \mathfrak{d}(\in \cap([0,1]\times\mathbb{K}));\\
\mathfrak{s} &= \mathfrak{b}(\text{LIM} \cap({}^\omega 2\times[\omega]^\omega)), & \mathfrak{r} &= \mathfrak{d}(\text{LIM} \cap({}^\omega 2\times[\omega]^\omega));\\
\mathfrak{s}_\sigma &= \mathfrak{b}(\text{LIM} \cap(\ell^\infty\times[\omega]^\omega)), & \mathfrak{r}_\sigma &= \mathfrak{d}(\text{LIM} \cap(\ell^\infty\times[\omega]^\omega));\\
\mathfrak{s} &= \mathfrak{d}(\chi\text{AOS} \cap([\omega]^\omega\times{}^\omega 2)), & \mathfrak{r} &= \mathfrak{b}(\chi\text{AOS} \cap([\omega]^\omega\times{}^\omega 2));\\
\mathfrak{s}_\sigma &= \mathfrak{d}(\chi\text{AOS} \cap([\omega]^\omega\times\ell^\infty)), & \mathfrak{r}_\sigma &= \mathfrak{b}(\chi\text{AOS} \cap([\omega]^\omega\times\ell^\infty));
\end{aligned}$$

PROOF. All invariants from [vD] and [Va] (on the left sides of inequalities) have their definitions in the same syntactical form as that of $\mathfrak{b}(R)$ and $\mathfrak{d}(R)$. The only exception beeing $\mathfrak{s}$ and $\mathfrak{r}$. But for $a \in {}^{\omega}2$ the $\lim_{n\in X} a(n)$ exists iff $a^{-1}(0) \supseteq^* X$ or $a^{-1}(1) \supseteq^* X$.

Notice that for the Boolean algebra $\mathcal{P}(\omega)/fin$ is $\mathfrak{b}(\supseteq^*) = 2$ and $\mathfrak{d}(\supseteq^*) = 2^\omega$ and so it is more interesting to study restricted versions of cardinal characteristics (so called boolean-like) such as $\mathfrak{p}, \mathfrak{t}, \mathfrak{h}, \mathfrak{u}$ and $\mathfrak{i}$ (see [Va]) . We do not consider this here. Numbers $\mathfrak{s}, \mathfrak{r}$ originaly defined as boolean-like appeared to be $\mathfrak{b}$ and $\mathfrak{d}$-like using the relation LIM.

3. Generalized Galois-Tukey connections.

3.1. The theory.

3.1.1. Definitions of connections, being simpler and of an algebraic category. In what follows we define the notion of a (generalized) Galois-Tukey connection (witnessed by maps). In 3.1.1.2. it is rephrased in the language of partial orders and in 3.1.1.3. in the language of algebraic theory of categories. This rephrasing (i.e. giving the same thing three different names) is important because we obtain in this way a motivation for asking questions which are natural for partial orders and for categories. All this together is summarized in 3.1.4. Consider two binary relations R and S.

3.1.1.1. Connections witnessed by maps. An ordered pair of functions (E, F) is called a <u>(generalized) Galois-Tukey connection</u> (also abbreviated as a GT-connection) between R and S if the following holds:

(a) $E : \mathrm{dom}(R) \longrightarrow \mathrm{dom}(S)$

(b) $F : \mathrm{rng}(S) \longrightarrow \mathrm{rng}(R)$

(c) $(\forall x \in \mathrm{dom}(R))(\forall v \in \mathrm{rng}(S))(E(x), v) \in S$ implies $(x, F(v)) \in R$.

3.1.1.2. The partial order of "being simpler than". The fact that there is a GT-connection between R and S is often rephrased as <u>"R is simpler than S"</u> (using a motivation of J. W. Tukey comming from convergence structures, [Tu]). The relation "to be simpler than" forms a partial order on the class of all binary relations (see also [Fr2]). The systematic study of this partial order suggests questions like: what are possible sizes of chains or antichains, minimal or maximal elements, predecessors or sucessors of a relation and/or restrict all this questions to, say Borel or Π^1_1 relations (see [Je]).

3.1.1.3. The category $\mathbb{GT}$. Using the language of the algebraic theory of categories (see [ML]) we can define the category $\mathbb{GT}$ as follows: <u>objects</u> are all binary relations R and a <u>morphism</u> $R \xrightarrow{\varphi} S$ is in fact the oriented tuple (R, S, E, F) if (E, F) witnesses the GT-connection between R and S. The systematic study of this category suggests questions like: characterize epi- and monomorphisms of $\mathbb{GT}$ or e.g. does $\mathbb{GT}$ have limits, colimits or admits quotients?; the study of certain subcategories e.g. relations of cardinality less or equal to continuum, Borel or Π^1_1 relations.

3.1.1.4. Further definitions and notation. The fact that R is simpler than S (or equivalently that there is a morphism from R to S) will be also denoted as $R \longrightarrow S$

in our diagrams in section 3.2. The mapping E (the first coordinate in the tuple (E, F) witnessing the connection) will be called the E-mapping of the connection and the mapping F will be called the F-mapping of the very connection.

3.1.1.5. The substantial part of R according to GT-equivalence. Assume $X_0 \subseteq \mathrm{dom}(R)$ and $Y_0 \subseteq \mathrm{rng}(R)$ and consider the relation $R_0 = R \cap (X_0 \times Y_0)$. We say that R_0 is a substantial part of R if R and R_0 are equivalent in the sense of generalized GT-connections in a very special way, that is that the F-mapping of $R \longrightarrow R_0$ and the E-mapping of $R_0 \longrightarrow R$ are identities. So for instance $\subseteq$ restricted to G_δ sets of measure zero is a substantial part of $\subseteq_{(\mathbb{L})^2}$. We will not complicate the topic and in the sequel we always assume we are working with some definable substantial part of the relation in question, especially in Section 3.2. and Capter 5.

OBSERVATION 3.1.2. *Assume R is simpler than S, then the following holds*

(1) $\mathfrak{b}(S) \leq \mathfrak{b}(R)$ *and* $\mathfrak{d}(S) \geq \mathfrak{d}(R)$
(2) $\neg S^{-1}$ *is simpler than* $\neg R^{-1}$.

PROOF. To see this just notice that if $E : \mathrm{dom}(R) \longrightarrow \mathrm{dom}(S)$ and $F : \mathrm{rng}(S) \longrightarrow \mathrm{rng}(R)$ are such that $E(x)Sv$ implies $xRF(v)$ then if $B \subseteq \mathrm{dom}(R)$ is an R-unbounded set then $E(B)$ is S-unbounded and if $D \subseteq \mathrm{rng}(S)$ is S-dominating the $F(D)$ is R-dominating. Moreover notice that "$E(x)Sv$ implies $xRF(v)$" is equivalent to "$F(v)\neg R^{-1}x$ implies $v\neg S^{-1}E(x)$".

COROLLARY 3.1.3. *If R is simpler than S then*

$$\mathfrak{d}(\neg S^{-1}) = \mathfrak{b}(S) \leq \mathfrak{b}(R) = \mathfrak{d}(\neg R^{-1})$$

and

$$\mathfrak{b}(\neg S^{-1}) = \mathfrak{d}(S) \geq \mathfrak{d}(R) = \mathfrak{b}(\neg R^{-1})$$

PROOF. It is a consequence of Observations 2.1.3. and 3.1.2.

In fact we used this in the introduction, when we showed that $\in \cap([0,1] \times \mathbb{L})$ is simpler than $\not\ni \cap(\mathbb{K} \times [0,1])$. This explains also why most results on cardinal invariants have two dual forms – the $\mathfrak{b}$-like and the $\mathfrak{d}$-like one (see item (3) in all assertions of Section 3.2.).

3.1.4. To summarize all the notions and graphical abbreviations and to explain the form of assertions in 3.2. we give the following:

SCHEMA. *Let R and S are binary relations. Denote*

(1) *–The relation $R \cap (X \times Y)$ is simpler than $S \cap (U \times V)$; or rephrased*
–there is a (generalized) Galois-Tukey connection between $R_{X\times Y}$ and $S_{U\times V}$; or rephrased
–in the category $\mathbb{GT}$ there is a morphism between $R_{X\times Y}$ and $S_{U\times V}$; or graphically
–$R_{X\times Y} \longrightarrow S_{U\times V}$.
(2) *There are mappings $E : X \longrightarrow U$ and $F : V \longrightarrow Y$ such that if $E(x)Sv$ then $xRF(v)$.*

(3) *$\mathfrak{b}(S_{U\times V}) \leq \mathfrak{b}(R_{X\times Y})$ and $\mathfrak{d}(S_{U\times V}) \geq \mathfrak{d}(R_{X\times Y})$; or graphically $\mathfrak{b}(S_{U\times V}) \longrightarrow \mathfrak{b}(R_{X\times Y})$ and $\mathfrak{d}(S_{U\times V}) \longleftarrow \mathfrak{d}(R_{X\times Y})$, here $\longrightarrow$ means $\leq$ is provable in ZFC.*

then any of notions mentioned in (1) is equivalent to (2) and (2) implies (3).

3.2. EXAMPLES.

In this section we give some groups of GT-connections which fit well together and we visualize this by graphical diagrams using conventions of 3.1.4.

Some of results have more authors: ones which proved inequalities between cardinal characteristics (as in 3.1.4.(3)) (these results are well available in literature) and those which observed constructions of E- and F-mappings (as in 3.1.4.(2)) (Great deal of this results come from [Fr1] and/or are unpublished). So in proofs we give mainly constructions of E- and F-mappings. The fact that they are well defined and fulfill the implication $E(x)Sv \implies xRF(v)$ can be (more or less) easily decoded from cited papers from proofs of inequalities between cardinal characteristics. Remind 3.1.1.5., without noticing we are working with some substantial part of relations in question.

3.2.1. Cichoń's diagram. We saw already that relations involved in cardinal characteristics forming the so called Cichoń's diagram (see [Fr1])are $\in$, $\subseteq$ and $\leq^*$. The following describes their connections.

THEOREM 3.2.1.1. *(T.Bartoszynski ([Ba]), J.Raisonier and J.Stern ([RaSt]), J.Cichoń, D.H.Fremlin and J.Pawlikowski (see [Fr1]))*

(1) *The relation $\subseteq \cap(\mathbb{K})^2$ is simpler than $\subseteq \cap(\mathbb{L})^2$*
(2) *or equivalently, there are mappings $E_1 : \mathbb{K} \longrightarrow \mathbb{L}$ and $F_1 : \mathbb{L} \longrightarrow \mathbb{K}$ such that if $E_1(X) \subseteq Y$ then $X \subseteq F_1(Y)$*
(3) *and consequently* $\mathrm{add}(\mathbb{L}) \leq \mathrm{add}(\mathbb{K})$ *and* $\mathrm{cof}(\mathbb{L}) \geq \mathrm{cof}(\mathbb{K})$.

PROOF. First we introduce some coding for structures and operations.
Coding the category:

(1) For $X \subseteq [0,1]$ let $\{U_n^X : n \in \omega\}$ enumerate a countable base for the relative topology of X;
(2) for $i \in \omega$ we construct $\mathcal{V}_i = \{V(i,j) : j \in \omega\}$ a countable family of open subsets of $U_i^{\mathcal{I}}$ such that
    (2.1) every open dense subset of $[0,1]$ includes some member of $\mathcal{V}_i$ and
    (2.2) $\bigcap_{r=0}^{(i+1)^2} V(i,j_r) \neq \emptyset$ for any sequence $j_0,\ldots,j_{(i+1)^2}$ of natural numbers;
(3) for $X \in \mathbb{K}$ let $\{H_i^X\}_{i\in\omega}$ be an increasing sequence of nowhere dense sets containing X.

Coding the measure:

(4) Let $\{G(i,j) : (i,j) \in \omega^2\}$ be an μ-independent double sequence of open subsets of $[0,1]$ such that $\mu(G(i,j)) = \frac{1}{(i+1)^2}$;

(5) for $Y \in \mathbb{L}$ take $K_Y \subseteq [0,1]\backslash Y$ compact such that for every U open $K_Y \cap U \neq \emptyset$ implies $\mu(K_Y \cap U) \geq 0$.

The heuristics is that acting of the double sequence $V(i,j)$ on category can be replaced by acting of double sequence $G(i,j)$ on measure (but the situation is not symetric). Also constructions of H_i^X and K_Y (decoded from a sequence K_i wittnessing a G_δ superset of Y of measure zero) are also not without similarity.
Definition of E_1. For $X \in \mathbb{K}$ put $g_X : \omega \longrightarrow \omega$ as follows:

$$g_X(i) = \min\{j : H_i^X \cap V(i,j) = \emptyset\}$$

and then

$$E_1(X) = \bigcap_{l=0}^{\infty} \bigcup_{i=l}^{\infty} G(i, g_X(i)).$$

Definition of F_1. For $Y \in \mathbb{L}$ put $h_Y : \omega \longrightarrow \omega$ as follows

$$h_Y(n) = \min\left\{k : \forall i \geq k \left|\{j : U_n^{K_Y} \cap G(i,j) = \emptyset\}\right| \leq \frac{(i+1)^2}{2^{n+1}}\right\}$$

and then

$$F_1(Y) = [0,1] \setminus \bigcap_{l=0}^{\infty} \bigcup_{i=l}^{\infty} \bigcap \{V(i,j) : \exists n \in \omega,\, i \geq h_Y(n) and\ G(i,j) \cap U_n^{K^Y} = \emptyset\}.$$

THEOREM 3.2.1.2. *(F.Rothberger ([Ro1]))*

(1) *The relation* $\in \cap([0,1] \times \mathbb{K})$ *is simpler than* $\not\ni \cap(\mathbb{L} \times [0,1])$
(2) *or equivalently, there are mappings* $E_2 : [0,1] \longrightarrow \mathbb{L}$ *and* $F_2 : [0,1] \longrightarrow \mathbb{K}$ *such that if* $E_2(x) \not\ni y$ *then* $x \in F_2(y)$
(3) *and consequently* $\mathrm{cov}(\mathbb{L}) \leq \mathrm{non}(\mathbb{K})$ *and* $\mathrm{non}(\mathbb{L}) \geq \mathrm{cov}(\mathbb{K})$.

PROOF. The construction of (2) is in introduction, put $E_2 = E'$ and $F_2 = F'$. So to have (3) by 3.1.2. we have

$$\mathrm{cov}(\mathbb{L}) = \mathfrak{d}(\in_{\mathcal{I}\times\mathbb{L}}) = \mathfrak{b}(\not\ni_{\mathbb{L}\times\mathcal{I}}) \leq \mathfrak{b}(\in_{\mathcal{I}\times\mathbb{K}}) = \mathrm{non}(\mathbb{K})$$

and

$$\mathrm{non}(\mathbb{L}) = \mathfrak{b}(\in_{\mathcal{I}\times\mathbb{L}}) = \mathfrak{d}(\not\ni_{\mathbb{L}\times\mathcal{I}}) \geq \mathfrak{d}(\in_{\mathcal{I}\times\mathbb{K}}) = \mathrm{cov}(\mathbb{K})$$

THEOREM 3.2.1.3.

(1) *The relation* $\not\geq^* \cap({}^\omega\omega)^2$ *is simpler than* $\leq^* \cap({}^\omega\omega)^2$
(3) *and consequently* $\mathfrak{b} \leq \mathfrak{d}$.

PROOF. Is easy, becuase it is true in general for partial orders that $x \leq y$ implies $x \not> y$.

THEOREM 3.2.1.4.

(1) *The relation* $\not\ni \cap(\mathbb{L} \times [0,1])$ *is simpler than* $\subseteq \cap(\mathbb{L})^2$
(2) *or equivalently, there are mappings* $E_3 : \mathbb{L} \longrightarrow \mathbb{L}$ *and* $F_3 : \mathbb{L} \longrightarrow [0,1]$ *such that if* $E_3(X) \subseteq Y$ *then* $X \not\ni F_3(Y)$
(3) *and consequently* $\mathrm{add}(\mathbb{L}) \leq \mathrm{cov}(\mathbb{L})$ *and* $\mathrm{cof}(\mathbb{L}) \geq \mathrm{non}(\mathbb{L})$.

PROOF. Is easy, take $E_3(X) = X$ and $F_3(Y) \in [0,1] \setminus Y$ arbitrary.

THEOREM 3.2.1.5. *(F.Rothberger ([Ro2]))*

(1) *The relation* $\in \cap([0,1] \times \mathbb{K})$ *is simpler than* $\leq^* \cap({}^\omega\omega)^2$
(2) *or equivalently, there are mappings* $E_4 : [0,1] \longrightarrow {}^\omega\omega$ *and* $F_4 : {}^\omega\omega \longrightarrow \mathbb{K}$ *such that if* $E_4(x) \leq^* f$ *then* $x \in F_4(f)$
(3) *and consequently* $\mathfrak{b} \leq \mathrm{non}(\mathbb{K})$ *and* $\mathfrak{d} \geq \mathrm{cov}(\mathbb{K})$.

PROOF. Fix $\varphi : [0,1] \longrightarrow {}^\omega\omega$ equal to a topological homeomorphism of irationals and ${}^\omega\omega$ (e.g. via continuous fractions) and put $E_4 = \varphi$. Define $F_4(f) = \{x \in [0,1] : \varphi(x) \leq^* f\}$.

THEOREM 3.2.1.6. *(J.Truss ([Tr]), A.W.Miller ([Mi1]))*

(1) *The relation* $\leq^* \cap({}^\omega\omega)^2$ *is simpler than* $\subseteq \cap(\mathbb{K})^2$
(2) *or equivalently, there are mappings* $E_5 : {}^\omega\omega \longrightarrow \mathbb{K}$ *and* $F_5 : \mathbb{K} \longrightarrow {}^\omega\omega$ *such that if* $E_5(f) \subseteq Y$ *then* $f \leq^* F_5(Y)$
(3) *and consequently* $\mathrm{add}(\mathbb{K}) \leq \mathfrak{b}$ *and* $\mathrm{cof}(\mathbb{K}) \geq \mathfrak{d}$.

PROOF. Similarly as in Lemma 3 of [Rep] define

$$E_5(f) = \left\{x \in {}^\omega 2 : (\exists m)(\forall n > m)(\exists i \leq \max_{j \leq n} f(j))(x(n+i) = 1)\right\}.$$

For defining F_5 we are working with reals represented by ${}^\omega 2$ with topology base consisting from sets with finite support. Recall (3) of 3.2.1.1. by seting $Y \subseteq \bigcup_{n \in \omega} H_n^Y$. Define

$$F_5(Y)(0) = 0$$
$$F_5(Y)(n+1) = \min\{j : \text{ there is some } I_n^Y \subseteq j \setminus F_5(Y)(n) \text{ such that } I_n^Y \text{ is a support of a basic open set disjoint with } H_n^Y\}.$$

3.2.1.7. Graphical form. To see the elegance of the emerging structure we displayed also all $\neg R^{-1}$ relations in question and where $R \longrightarrow S$ holds also $\neg S^{-1} \longrightarrow \neg R^{-1}$ is displayed (note that $\longrightarrow$ means "is simpler", see 3.1.4.)

$$\begin{array}{ccccccc}
\in_{\mathcal{I}\times\mathbb{L}} & \longrightarrow & \not\ni_{\mathbb{K}\times\mathcal{I}} & \longrightarrow & \subseteq_{(\mathbb{K})^2} & \longrightarrow & \subseteq_{(\mathbb{L})^2} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \not\leq^*_{({}^\omega\omega)^2} & \longrightarrow & \leq^*_{({}^\omega\omega)^2} & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\not\supseteq_{(\mathbb{L})^2} & \longrightarrow & \not\supseteq_{(\mathbb{K})^2} & \longrightarrow & \in_{\mathcal{I}\times\mathbb{K}} & \longrightarrow & \not\ni_{\mathbb{L}\times\mathcal{I}}
\end{array}$$

and it is probably not surprising that the shape is the same as that of Cichoń's diagram ([Fr1]) (here the arrow $\longrightarrow$ means "$\leq$ is provable in ZFC")

$$
\begin{array}{ccccccc}
\mathrm{cov}(\mathbb{L}) & \longrightarrow & \mathrm{non}(\mathbb{K}) & \longrightarrow & \mathrm{cof}(\mathbb{K}) & \longrightarrow & \mathrm{cof}(\mathbb{L}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{add}(\mathbb{L}) & \longrightarrow & \mathrm{add}(\mathbb{K}) & \longrightarrow & \mathrm{cov}(\mathbb{K}) & \longrightarrow & \mathrm{non}(\mathbb{L})
\end{array}
$$

Sensitive to phenomena of similarity in the nature we note that in general $\in$ is simpler than $\subseteq$, $\not\ni$ is simpler than $\subseteq$ and $\in$ is simpler than $\not\ni$ (when restricted to the same field) .

3.2.2. The existence of limit. We investigated different forms of the existence of limits in [Vo1], [Vo2] and [Vo3]. Here are the connections. It is our choice to give them in LIM-form or in the χAOS-form. As they fit well together with the middle row of Cichoń's diagram, we show them together. Some connections are trivial because identical mappings or embeddings between structures wittness the connection. Moreover $[\omega]^\omega$ can be considered as a special case of diagonal matrices: identify $X \in [\omega]^\omega$ with the matrix $C_X = \{c(n,i) : n, i \in \omega\}$ where $c(n,i) = 1$ if and only if i is the n-th element of X, else $c(n,i) = 0$. Note that for arbitrary $x_n \in {}^\omega\mathcal{R}$ then $\lim_{n\in X} x_n = \mathrm{C_X lim}_{n\to+\infty} x_n$. The nontrivial connections are the following:

THEOREM 3.2.2.1.

(1) *The relation* $\chi\mathrm{AOS} \cap ([\omega]^\omega \times {}^\omega 2)$ *is simpler than* $\not\ni \cap (\mathbb{L} \times [0,1])$
(2) *or equivalently, there are mappings* $E_6 : [\omega]^\omega \longrightarrow \mathbb{L}$ *and* $F_6 : [0,1] \longrightarrow {}^\omega 2$ *such that if* $E_6(X) \not\ni x$ *then* $\lim_{n\in X} F_6(x)(n)$ *does not exist*
(3) *and consequently* $\mathrm{cov}(\mathbb{L}) \leq \mathfrak{r}$ *and* $\mathrm{non}(\mathbb{L}) \geq \mathfrak{s}$.

PROOF. Similarly as in [Vo1] we represent $\mathbb{L}$ for the product measure on ${}^\omega 2$ and define $E_6(X) = \{f \in {}^\omega 2 : f^{-1}(0) \supseteq^* X \text{ or } f^{-1}(1) \supseteq^* X\}$ and $F_6(x) = x$ (via diadic expansion).

THEOREM 3.2.2.2.

(1) *The relation* $\mathrm{R}\chi\mathrm{AOS} \cap (\mathbb{M} \times {}^\omega 2)$ *is simpler than* $\not\ni \cap (\mathbb{K} \times [0,1])$
(2) *or equivalently, there are mappings* $E_7 : \mathbb{M} \longrightarrow \mathbb{K}$ *and* $F_7 : [0,1] \longrightarrow {}^\omega 2$ *such that if* $E_7(C) \not\ni x$ *then* $\mathrm{Clim}_{n\to+\infty} F_7(x)(n)$ *does not exist*
(3) *and consequently* $\mathrm{cov}(\mathbb{K}) \leq \mathfrak{b}(\mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^\omega 2})$ *and* $\mathrm{non}(\mathbb{K}) \geq \mathfrak{d}(\mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^\omega 2})$.

PROOF. As in [Vo2] put for a regular matrix C, $E_7(C) = \{x \in [0,1] : \mathrm{Clim}_{n\to+\infty} x_n \text{ exists }\}$ and $F_7(x) = x$.

THEOREM 3.2.2.3.

(1) *The relation* $\mathrm{R}\chi\mathrm{AOS} \cap (\mathbb{M} \times {}^\omega 2)$ *is simpler than* $\leq^* \cap ({}^\omega\omega)^2$

(2) *or equivalently, there are mappings* $E_8 : \mathbb{M} \longrightarrow {}^{\omega}\omega$ *and* $F_8 : {}^{\omega}\omega \longrightarrow {}^{\omega}2$ *such that if* $E_8(C) \leq^* f$ *then* $\operatorname{Clim}_{n\to+\infty} F_8(f)(n)$ *does not exist.*

(3) *and consequently* $\mathfrak{b} \leq \mathfrak{b}(\mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^{\omega}2})$ *and* $\mathfrak{d} \geq \mathfrak{d}(\mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^{\omega}2})$.

Proof. Similarly as in Theorem 3.2 of [Vo2] and Theorem 2 of [Vo3] for a regular matrix $C = \{c(n,i) : (n,i) \in \omega^2\}$ define (for all but finitely many cases)

$$l_n = \sup\left\{k : \sum_{i=0}^{k} c(n,i) < \frac{1}{8}\right\} \quad \text{and} \quad r_n = \min\left\{k : \sum_{i=k}^{\infty} c(n,i) < \frac{1}{8}\right\}.$$

Then $E_8(C)(n+1) = \min\left\{j : l_j > r_{E_8(C)(n)}\right\}$.

For $f \in {}^{\omega}\omega$ denote by $\overline{f}$ the iteration $\overline{f}(0) = f(0)+1$ and $\overline{f}(n+1) = f(\overline{f}(n)+n)$, and define

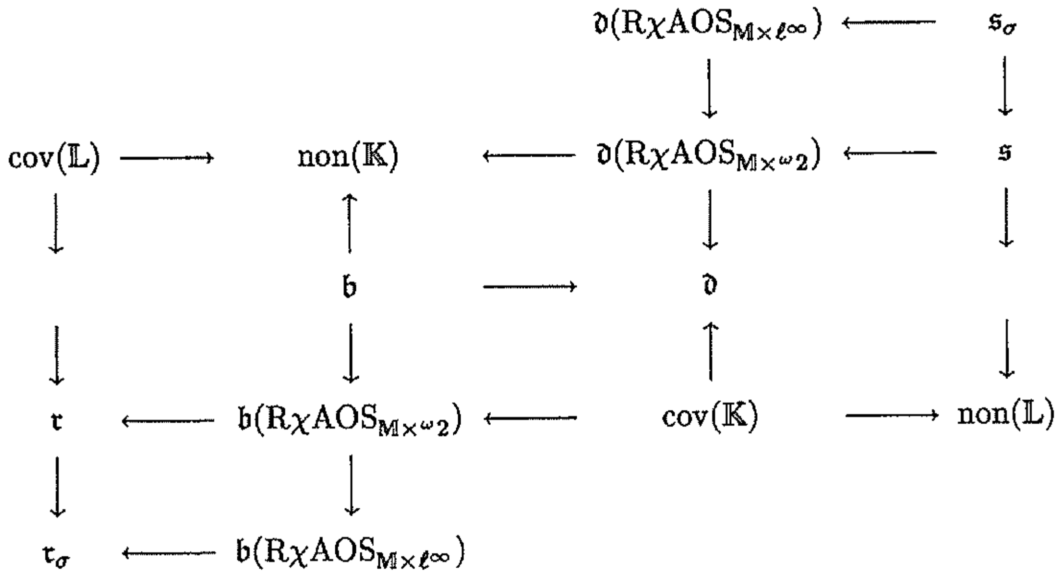
$$F_8(f)(i) = \begin{cases} 0, & \text{if } i \in [\overline{f}(4n), \overline{f}(4n+2)) \text{ for some } n \in \omega, \\ 1, & \text{if } i \in [\overline{f}(4n+2), \overline{f}(4n+4)) \text{ for some } n \in \omega. \end{cases}$$

In cited papers it is actually proven (consistently) more, namely $\mathfrak{d}(\mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^{\omega}2}) \leq \mathfrak{bs}$ and $\mathfrak{b}(\mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^{\omega}2}) \geq \min(\mathfrak{d}, \mathfrak{t})$. We are not able to carry out these inequalities through GT-connections. This topic will be discussed in Chapter 4.

3.2.2.4. The graphical form of connections between various forms of limits and/or chaos

$$\begin{array}{ccccccc}
 & & & & \mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times\ell^\infty} & \longleftarrow & \chi\mathrm{AOS}_{[\omega]^\omega\times\ell^\infty} \\
 & & & & \downarrow & & \downarrow \\
\in_{\mathcal{I}\times\mathbb{L}} & \longrightarrow & \not\ni_{\mathbb{K}\times\mathcal{I}} & \longleftarrow & \mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^{\omega}2} & \longleftarrow & \chi\mathrm{AOS}_{[\omega]^\omega\times{}^{\omega}2} \\
\downarrow & & \uparrow & & \downarrow & & \downarrow \\
 & & \not\geq^*_{({}^{\omega}\omega)^2} & \longrightarrow & \leq^*_{({}^{\omega}\omega)^2} & & \\
\downarrow & & \downarrow & & \uparrow & & \downarrow \\
\mathrm{LIM}_{{}^{\omega}2\times[\omega]^\omega} & \longleftarrow & \mathrm{RLIM}_{{}^{\omega}2\times\mathbb{M}} & \longleftarrow & \in_{\mathcal{I}\times\mathbb{K}} & \longrightarrow & \not\ni_{\mathbb{L}\times\mathcal{I}} \\
\downarrow & & \downarrow & & & & \\
\mathrm{LIM}_{\ell^\infty\times[\omega]^\omega} & \longleftarrow & \mathrm{RLIM}_{\ell^\infty\times\mathbb{M}} & & & &
\end{array}$$

and corresponding consequences for cardinal characteristics ($\longrightarrow$ means $\leq$ is provable in ZFC) are:



3.2.3. Absolute summability. There are several results indicating that the relations connected with the absolute convergence or divergence of series (involved in the relation itself or in the domain we restrict to) are of certain interest. Moreover for the first time we come in contact with relations which are partial orders and are not upwards directed and which therefore generate Boolean algebras. Surprisingly they are laying all very high in our diagrams (we mean $\precsim^*_{(\ell^1_+)^2}$, $\supseteq^*_{([\omega]^\omega)^2}$ and $\geq^*_{(\ell^\infty\setminus\ell^1)^2}$). Remind 2.1.2. and Notice in 2.2.2.

THEOREM 3.2.3.1.

(1) *The relation* $\leq^* \cap ({}^\omega\omega)^2$ *is simpler than* $\mathrm{CONV} \cap (c_0 \times [\omega]^\omega)$
(2) *or equivalently, there are mappings* $E_9 : {}^\omega\omega \longrightarrow c_0 \setminus \ell^1$ *and* $F_9 : [\omega]^\omega \longrightarrow {}^\omega\omega$ *such that if* $\sum_{n\in X} |E_9(f)(n)| < +\infty$ *then* $f \leq^* F_9(X)$
(3) *and consequently* $\mathfrak{b}(\mathrm{CONV}_{c_0\times[\omega]^\omega}) \leq \mathfrak{b}$ *and* $\mathfrak{d}(\mathrm{CONV}_{c_0\times[\omega]^\omega}) \geq \mathfrak{d}$.

PROOF. Define (see [Vo1]) $E_9(f)(i) = \frac{\log(k+1)}{k+1}$ if $i \in (f(k-1), f(k)|$ and $F_9(X)(n) = \min\{i : |X \cap (i+1)| = n\}$ i.e. the enumeration function of X.

THEOREM 3.2.3.2.

(1) *The relation* $\mathrm{CONV} \cap (c_0 \times [\omega]^\omega)$ *is simpler than* $\leq^* \cap ({}^\omega\omega)^2$
(2) *or equivalently, there are mappings* $E_{10} : c_0 \longrightarrow {}^\omega\omega$ *and* $F_{10} : {}^\omega\omega \longrightarrow [\omega]^\omega$ *such that if* $E_{10}(a) \leq^* f$ *then* $\sum_{n\in F_{10}(f)} |a(n)| < +\infty$
(3) *and consequently* $\mathfrak{b} \leq \mathfrak{b}(\mathrm{CONV}_{c_0\times[\omega]^\omega})$ *and* $\mathfrak{d} \geq \mathfrak{d}(\mathrm{CONV}_{c_0\times[\omega]^\omega})$.

PROOF.

$$E_{10}(a)(k) = \min\left\{i : (\forall j > i)(|a(j)| < \frac{1}{2^k})\right\} \quad \text{and} \quad F_{10}(f) = \mathrm{rng}(f).$$

Both results 3.2.3.1. and 3.2.3.2. come from [Vo1] and they show in fact that $\leq^*_{({}^\omega\omega)^2}$ and $\mathrm{CONV}_{c_0\times[\omega]^\omega}$ are equivalent according to Galois-Tukey connections, and moreover

$$\leq^*_{({}^\omega\omega)^2} \longleftrightarrow \mathrm{CONV}_{c_0\times[\omega]^\omega} \longrightarrow \mathrm{CONV}_{h_0\times[\omega]^\omega}$$

holds (we do not display this in 3.2.3.7.).

THEOREM 3.2.3.3.

(1) *The relation* $\mathrm{CONV} \cap (h_0 \times [\omega]^\omega)$ *is simpler than* $\supseteq^*_{([\omega]^\omega)^2}$
(2) *or equivalently, there are mappings* $E_{11} : h_0 \longrightarrow [\omega]^\omega$ *and* $F_{11} : [\omega]^\omega \longrightarrow [\omega]^\omega$ *such that if* $E_{11}(a) \supseteq^* X$ *then* $\sum_{n\in F_{11}(X)} |a(n)| < +\infty$

PROOF. Put first $E'_{11}(a)(n) = \min\left\{i : |a(i)| < \frac{1}{2^n}\right\}$ and then define $E_{11}(a) = \left\{E'_{11}(a)(n) : n \in \omega\right\} = \mathrm{rng}(E'_{11}(a))$, $F_{11}(X) = X$.

THEOREM 3.2.3.4.

(1) *The relation* $\mathrm{CONV} \cap (h_0 \times [\omega]^\omega)$ *is simpler than* $\geq^* \cap (\ell^\infty \setminus \ell^1)^2$
(2) *or equivalently, there are mappings* $E_{12} : h_0 \setminus \ell^1 \longrightarrow \ell^\infty \setminus \ell^1$ *and* $F_{12} : \ell^\infty \setminus \ell^1 \longrightarrow [\omega]^\omega$ *such that*

$$\textit{if } E_{12}(a) \geq^* b \textit{ then } \sum_{n\in F_{12}(b)} |a(n)| < +\infty.$$

PROOF.

$$E_{12}(a)(n) = \begin{cases} 0, & \text{if } n \notin \mathrm{rng}(E'_{11}(a)) = E_{11}(a), \\ \frac{1}{k}, & \text{if } n = E'_{11}(k). \end{cases}$$

THEOREM 3.2.3.5. *(T.Bartoszynski([Ba]))*

(1) *The relation* $\subseteq \cap (\mathbb{L})^2$ *is simpler than* $\leq^* \cap (\ell^1)^2$
(2) *or equivalently, there are mappings* $E_{13} : \mathbb{L} \longrightarrow \ell^1$ *and* $F_{13} : \ell^1 \longrightarrow \mathbb{L}$ *such that if* $E_{13}(X) \leq^* a$ *then* $X \subseteq F_{13}(a)$
(3) *([Ba])* $\mathrm{add}(\mathbb{L}) = \mathfrak{b}(\leq^*_{(\ell^1)^2})$ *and* $\mathrm{cof}(\mathbb{L}) = \mathfrak{d}(\leq^*_{(\ell^1)^2})$.

PROOF. Fix a base $\mathcal{U} = \{U_n : n \in \omega\}$ of topology on $[0,1]$ and find for every $X \in \mathbb{L}$ an $f_X \in {}^\omega 2$ such that $A_X = \bigcap_{n=0}^{\infty} \bigcup \{U_m : m \geq n \textit{ and } f_X(m) = 1\}$ is of measure zero and $X \subseteq A_X$. Define

$$E_{13}(X)(n) = f_X(n)\mu(U_n)$$

and

$$F_{13}(a) = \bigcap_{n=0}^{\infty} \bigcup \{U_m : m \geq n \textit{ and } |a(m)| \geq \mu(U_m)\}.$$

For stronger (3) see 4.2.2.3.

3.2.3.6. Easily the relation $\leq^* \cap (\ell^1)^2$ is simpler than $\precsim^* \cap (\ell^1_+)^2$.

3.2.3.7. The diagram of absolute convergence and/or divergence .

$$\begin{array}{ccccc}
\subseteq_{(\mathbb{L})^2} & \longrightarrow & \leq^*_{(\ell^1)^2} & \longrightarrow & \precsim^*_{(\ell^1_+)^2} \\
\uparrow & & & & \\
\subseteq_{(\mathbb{K})^2} & & \supseteq^*_{([\omega]^\omega)^2} & & \\
\uparrow & & \uparrow & & \\
\leq^*_{({}^\omega\omega)^2} & \longrightarrow & \mathrm{CONV}_{h_0 \times [\omega]^\omega} & \longrightarrow & \geq^*_{(\ell^\infty \setminus \ell^1)^2}
\end{array}$$

3.2.4. Some variations of previous relations and asymptotic density. In [Fr2] is in our language proven the following (which we therefore present just by diagram together with the relevant part of 3.2.1.7.):

$$\begin{array}{ccccccc}
\leq^*_{({}^\omega\omega)^2} & \longrightarrow & \leq_{({}^\omega\omega)^2} & \longrightarrow & \subseteq_{(\mathfrak{E})^2} & \longrightarrow & \subseteq_{(\mathfrak{F})^2} \\
\downarrow & & & & \downarrow & & \downarrow \\
 & & & & \subseteq_{(\mathfrak{Z})^2} & \longrightarrow & \leq_{(\ell^1)^2} \\
\downarrow & & & & & & \uparrow \\
\subseteq_{(\mathbb{K})^2} & \longrightarrow & \subseteq_{(\mathbb{L})^2} & \longrightarrow & & \longrightarrow & \leq^*_{(\ell^1)^2}
\end{array}$$

4. Handling max and min, product in the category $\mathbb{GT}$.

4.1. Products and coproducts in general.

Proofs of most of nontrivial results between max and min of cardinal characteristics (probably first was that of A.W.Miller, see 4.2.2.1.) have the following form

THEOREM 4.1.1. *Consider binary relations R, S and T and assume that there are mappings*

(1) $E_1 : \mathrm{dom}(R) \longrightarrow \mathrm{dom}(T)$
(2) $E_2 : \mathrm{dom}(R) \times \mathrm{rng}(T) \longrightarrow \mathrm{dom}(S)$
(3) $F : \mathrm{rng}(S) \times \mathrm{rng}(T) \longrightarrow \mathrm{rng}(R)$

such that $(\forall x \in \mathrm{dom}(R))(\forall z \in \mathrm{rng}(T))(\forall v \in \mathrm{rng}(S))$ *the following holds*

$$E_1(x) T z \textit{and } E_2(x,z) S v \implies x R F(z,v).$$

Then

$$\min(\mathfrak{b}(T), \mathfrak{b}(S)) \leq \mathfrak{b}(R)$$

and

$$\max(\mathfrak{d}(T), \mathfrak{d}(S)) \geq \mathfrak{d}(R)$$

PROOF. Straightforward

The question is how does this fit into our theory - or, what is the right definition of a relation (defined from T and S, if there is any) such that R should be simpler than the very relation we look for. (The heuristics says that the bigger $\mathfrak{b}(R)$ indicates that

R should be simpler than "something defined from S and T"). The straightforward idea is to take a product.

4.1.2. Definition of $T \times S$. (see [Vo5]) Consider two binary relations T and S and define

$$T \times S = \{((a,\epsilon),(z,w)) : aTz and\ \epsilon = 0 \text{ or } aSw and\ \epsilon = 1\}$$

OBSERVATION 4.1.3.

$$\mathfrak{b}((T \times S)) = \min(\mathfrak{b}(T), \mathfrak{b}(S))$$
$$\mathfrak{d}((T \times S)) = \max(\mathfrak{d}(T), \mathfrak{d}(S))$$

and both T and S are simpler than $T \times S$.

4.1.4. Problem. Is it true, that if R, S and T are as in the max-min diagram of 4.1.1. then R is simpler than $T \times S$?

We discuss some special cases of this problem in 4.2. (namely Problems 4.2.2.1.-4.).

We now develop a notion of product and coproduct in the category $\mathbb{GT}$.

4.1.5. Definition. Assume R is a binary relation. R is said to be directed, if for arbitrary finite $X \subseteq \mathrm{dom}(R)$ there is a single $y \in \mathrm{rng}(R)$ such that $(\forall x \in X)(xRy)$ (i.e. when $\mathfrak{b}(R) \geq \aleph_0$).

4.1.6. The coproduct in the category $\mathbb{GT}$. Assume R, S and T are binary relations such that R is simpler than T and S is simpler than T too. If moreover T is directed, then $R \times S$ is simpler than T. In graphical form:

$$\begin{array}{ccccc} & & T & & \\ & & \uparrow & & \\ R & \longrightarrow & R \times S & \longleftarrow & S \end{array}$$

So in terms of the algebraic theory of categories $R \times S$ is a coproduct in the subcategory of directed relations (see [ML]).

4.1.7. The product in the category $\mathbb{GT}$. For two binary relations P_1 and P_2 define

$$P_1 \odot P_2 = \neg(\neg P_2^{-1} \times \neg P_1^{-1})^{-1}.$$

Then

$$\mathfrak{b}(P_1 \odot P_2) = \max(\mathfrak{b}(P_1), \mathfrak{b}(P_2)) \text{ and } \mathfrak{d}(P_1 \odot P_2) = \min(\mathfrak{d}(P_1), \mathfrak{d}(P_2)).$$

If moreover Q is such that $Q \longrightarrow P_1$ and $Q \longrightarrow P_2$ and $\neg Q^{-1}$ is directed then

$$\begin{array}{ccccc} & & Q & & \\ & & \downarrow & & \\ P_2 & \longleftarrow & P_1 \odot P_2 & \longrightarrow & P_1 \end{array}$$

So $P_1 \odot P_2$ is the product in the category of binary relations with $\mathfrak{d}(Q) \geq \aleph_0$ (see [ML]).

4.2. FOUR EXAMPLES AND PROBLEMS.

4.2.1. Expanded Cichoń's diagram. Properties of products and coproducts and of previous sections allows us to present following:

$$
\begin{array}{ccccccc}
\in_{\mathcal{I}\times\mathbb{L}} & \longrightarrow & \not\ni_{\mathbb{K}\times\mathcal{I}} & \longrightarrow & \subseteq_{(\mathbb{K})^2} & \longrightarrow & \subseteq_{(\mathbb{L})^2} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^{\omega}2}\times\not\geq^*_{({}^{\omega}\omega)^2} & \longrightarrow & \leq^*_{({}^{\omega}\omega)^2} & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \chi\mathrm{AOS}_{[\omega]^{\omega}\times\ell^{\infty}}\times\not\geq^*_{({}^{\omega}\omega)^2} & & \mathrm{LIM}_{\ell^{\infty}\times[\omega]^{\omega}}\odot\leq^*_{({}^{\omega}\omega)^2} & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\not\supseteq_{(\mathbb{L})^2} & \longrightarrow & \not\supseteq_{(\mathbb{K})^2} & \longrightarrow & \in_{\mathcal{I}\times\mathbb{K}} & \longrightarrow & \not\ni_{\mathbb{L}\times\mathcal{I}}
\end{array}
$$

PROOF. As for any binary relations R, S and T we have $T \longrightarrow T\times S$ and if $R \longrightarrow S$ then also $R\times T \longrightarrow S\times T$ (and dually for $\odot$) the only we should explain are following: By 3.2.2.3. $\mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^{\omega}2}$ is simpler than $\leq^*_{({}^{\omega}\omega)^2}$ and by 3.2.2.2. is the same $\mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^{\omega}2}$ simpler than $\not\ni_{\mathbb{K}\times\mathcal{I}}$ and moreover both $\not\ni_{\mathbb{K}\times\mathcal{I}}$ and $\leq^*_{({}^{\omega}\omega)^2}$ are directed so by 4.1.6. we have $\mathrm{R}\chi\mathrm{AOS}_{\mathbb{M}\times{}^{\omega}2}\times\not\geq^*_{({}^{\omega}\omega)^2}$ is simpler than each of $\leq^*_{({}^{\omega}\omega)^2}$ and $\not\ni_{\mathbb{K}\times\mathcal{I}}$. Using 4.1.7. we obtain remaining dual connections.

In the graphical form we display a weaker form of the corresponding generalized Cichon's diagram consisting of cardinal characteristics from [Va] which even colapses to a simplier one because $\mathfrak{s} = \mathfrak{s}_\sigma$ ([Vol]) and the following trivial $\min(\mathfrak{d},\mathfrak{r}) \leq \min(\mathfrak{d}\mathfrak{r}_\sigma) \leq \mathfrak{d}$, which we do not display, so

$$
\begin{array}{ccccccc}
\mathrm{cov}(\mathbb{L}) & \longrightarrow & \mathrm{non}(\mathbb{K}) & \longrightarrow & \mathrm{cof}(\mathbb{K}) & \longrightarrow & \mathrm{cof}(\mathbb{L}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathfrak{b}\mathfrak{s} & \longrightarrow & \mathfrak{d} & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \mathfrak{b} & \longrightarrow & \min(\mathfrak{d},\mathfrak{r}) & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathrm{add}(\mathbb{L}) & \longrightarrow & \mathrm{add}(\mathbb{K}) & \longrightarrow & \mathrm{cov}(\mathbb{K}) & \longrightarrow & \mathrm{non}(\mathbb{L})
\end{array}
$$

4.2.2. Special cases of $T\times S$ in diagrams of 3.2. In this section we consider examples of relations R and $S\times T$ as in 4.1.4. and/or 4.1.1. such that the connection between R and $S\times T$ is possible as far as we know that the cardinal characteristics fulfils necessary inequalities.

4.2.2.1. Cichoń's diagram. According to 4.1.6. we know that following connections exist:

$$\begin{array}{ccccc} & & \subseteq_{(\mathbb{K})^2} & & \\ & & \uparrow & & \\ \not\ni_{\mathbb{K}\times\mathcal{I}} & \longrightarrow & (\not\ni_{\mathbb{K}\times\mathcal{I}}) \times (\leq^*_{({}^\omega\omega)^2}) & \longleftarrow & \leq^*_{({}^\omega\omega)^2} \end{array}$$

A theorem of A.W.Miller (see [Mil]) states that $\mathrm{add}(\mathbb{K}) = \min(\mathfrak{b}, \mathrm{cov}(\mathbb{K}))$ and $\mathrm{cof}(\mathbb{K}) = \max(\mathfrak{d}, \mathrm{non}(\mathbb{K}))$, so the following is reasonable:

Problem. Is $\subseteq_{(\mathbb{K})^2}$ simpler than $\not\ni_{\mathbb{K}\times\mathcal{I}} \times \leq^*_{({}^\omega\omega)^2}$? (And consequently equivalent.)

4.2.2.2. The existence of limits. In [Vo2] and [Vo3] it is proved that $\mathfrak{d}(\mathrm{R}\chi\mathrm{AOS}_{\mathrm{M}\times{}^\omega 2}) \leq \mathfrak{bs}$ and $\mathfrak{b}(\mathrm{R}\chi\mathrm{AOS}_{\mathrm{M}\times{}^\omega 2}) \geq \min(\mathfrak{d}, \mathfrak{t})$. The corresponding situation which visualizes the position of the very product in the diagram is the following:

$$\begin{array}{ccccc} \not\geq^*_{({}^\omega\omega)^2} & \longrightarrow & (\not\geq^*_{({}^\omega\omega)^2}) \times \chi\mathrm{AOS}_{[\omega]^\omega\times{}^\omega 2} & \longrightarrow & \leq^*_{({}^\omega\omega)^2} \\ & & \uparrow & & \uparrow \\ & & \chi\mathrm{AOS}_{[\omega]^\omega\times{}^\omega 2} & \longrightarrow & \mathrm{R}\chi\mathrm{AOS}_{\mathrm{M}\times{}^\omega 2} \end{array}$$

and the following is open.

Problem. Is $\mathrm{R}\chi\mathrm{AOS}_{\mathrm{M}\times{}^\omega 2}$ simpler than $(\not\geq^*_{({}^\omega\omega)^2}) \times (\chi\mathrm{AOS}_{[\omega]^\omega\times{}^\omega 2})$?

4.2.2.3. We just mention a problem of a similar kind without discussing it in detail. In [Ba] T.Bartoszynski proved that $\mathfrak{b}(\leq^*_{(\ell^1)^2}) = \mathrm{add}(\mathbb{L})$ and $\mathfrak{d}(\leq^*_{(\ell^1)^2}) = \mathrm{cof}(\mathbb{L})$ and the following is open:

Problem. Is $\leq^*_{(\ell^1)^2}$ simpler than $\subseteq_{(\mathbb{L})^2}$?

4.2.2.4. Last example in which situation of 4.1.1. occured is in [Fr3], namely: there are E_1, E_2 and F for $R = \leq_{(\ell^1)^2}$, $S = \subseteq_{(3)^2}$ and $T = \leq_{({}^\omega\omega)^2}$ (and also (another) for $R = \leq^*_{(\ell^1)^2}$, $S = \subseteq^*_{(3)^2}$ and $T = \leq^*_{({}^\omega\omega)^2}$) which fulfil properties of max-min diagram of 4.1.1. We do not know if the following is true

Problem ([Fr3]). Is $\leq^*_{(\ell^1)^2}$ simpler than $\leq^*_{({}^\omega\omega)^2} \times \subseteq^*_{(3)^2}$?

5. Absoluteness properties of $\mathbb{GT}$ connections.

In this section we consider the set-theoretic notion of absoluteness between two models of set theory $M \subseteq N$. The point is that absolute E-mappings and/or absolute F-mappings transfer some properties of generic extensions.

5.1. The concept of R^M-unbounded and R^M-dominating object.

Consider two models of set theory $M \subseteq N$. All structures we are working with can be considered to be definable (or they have a substantial part which is definable - see 5.1.1.) and so for any set X and relation R it makes sense to speak about X^M and R^M (respectively X^N and R^N) which are interpretations of definitions of X and R inside the model M (or inside N resp.). These interpretations can be carried out along the cumulative hierarchy of sets $V = \bigcup V_\alpha$ (see [Je]) but for certain substantial parts of structures of real analysis there has been developed the coding machinery of descriptive set theory (see [Je]) which carries most objects coded by reals (which are in the first level of the cumulative hierarchy over M).

For a real x denote by B_x the very Borel set with code x (B_x^M relativized to a model M). The set of all codes of some class $\mathcal{D}$ of Borel hierarchy we denote by $C_{\mathcal{D}}$ (see [Je]).

5.1.1. The substantial part of R according to GT-equivalence. Recall 3.1.1.5., we are working with some substantial part of R when necessary, this will be important in this Chapter.

5.1.2. Definition. Consider two models $M \subseteq N$ of set theory and a definable binary relation R. We assume moreover that all relations are absolute i.e. (e.g. for R) for $x, y \in M$, xR^My iff $x^NR^Ny^N$. (Note that all explicit relations in our paper are absolute.) Then $x \in \text{dom}(R^N)$ is said to be an R^M-unbounded object if for all $y \in \text{rng}(R^M)$ it is not the case that xR^Ny^N. An $y \in \text{rng}(R^N)$ is said to be an R^M-dominating object if for all $x \in \text{dom}(R^M)$ we have x^NR^Ny. (Our convention from 2.1.2. avoids trivial cases.)

Sometimes properties of generic extensions are investigated concerning the question whether $\text{dom}(R^M)^N$ is R-unbounded in N and whether $\text{rng}(R^M)^N$ is R-dominating in N. Observe that

(a) $\{x^N : x \in \text{dom}(R^M)\} = \text{dom}(R^M)^N$ is R-unbounded set in N iff there are no R^M-dominating objects in N

and

(b) $\{x^N : x \in \text{rng}(R^M)\} = \text{rng}(R^M)^N$ is R-dominating set in N iff there are no R^M-unbounded objects in N.

Much more difficult is the question: which forcing adds R^M-unbounded resp. R^M-dominating objects. Compare with [BaJuSh], [JuSh], [Mil] and [BuRecRep].

5.1.3. Definition of absolute mappings. A definable mapping $f : X \longrightarrow Y$ is absolute according to models $M \subseteq N$ if

$$(\forall x \in X^M)(f^N(x^N) = (f^M(x))^N.$$

That is: if f as interpreted in N acts on objects with codes in M, it gives the same result as when interpreted in M.

5.1.4. Absolute E-mappings. Assume $M \subseteq N$ and R is simpler than S witnessed by (E, F) and moreover E is absolute according to M and N. Then if there is an S^M-dominating object then there is a R^M-dominating object. So the existence of dominating objects is preserved in the opposite direction as absolute E-mappings are going. Of course equivalently, under the same assumptions if $\text{dom}(R^M)^N$ is an R-unbounded set in N then $\text{dom}(S^M)^N$ is an S-unbounded set in N.

5.1.5. Absolute F-mappings. Analogously consider $M \subseteq N$ and R is simpler than S witnessed by (E, F) and moreover F is absolute according to $M \subseteq N$. Then if there is an R^M-unbounded object then there is an S^M-unbounded object; and similarly $\text{rng}(S^M)^N$ being S-dominating set implies $\text{rng}(R^M)^N$ is R-dominating.

5.1.6. Observe moreover that

$$\text{an } R^M\text{-unbounded object is an } (\neg R^{-1})^M\text{-dominating object}$$

and

$$\text{an } R^M\text{-dominating object is an } (\neg R^{-1})^M\text{-unbounded object.}$$

Using preservation properties of 5.1.4., 5.1.5. and 5.1.6. we can derive lot of informations about generic extensions (having some basic informations, see e.g. 5.3.1. and 5.3.2.).

5.1.7. Galois-Tukey mappings acting on codes. As we are not able to carry out all mappings to be absolute we define some of them acting on codes.

Assume R and S are definable binary relations with (substantial part) having domains and ranges consisting of Borel sets. Consider

$$C_{\mathrm{dom}(R)},\ C_{\mathrm{rng}(R)},\ C_{\mathrm{dom}(S)} \text{ and } C_{\mathrm{rng}(S)}$$

sets of all Borel codes of elements of

$$\mathrm{dom}(R),\ \mathrm{rng}(R),\ \mathrm{dom}(S) \text{ and } \mathrm{rng}(S).$$

Let (E, F) be a pair of mappings

(1) $E : C_{\mathrm{dom}(R)} \longrightarrow C_{\mathrm{dom}(S)}$ and
(2) $F : C_{\mathrm{rng}(S)} \longrightarrow C_{\mathrm{rng}(R)}$

such that

$$(\forall x \in C_{\mathrm{dom}(R)})(\forall v \in C_{\mathrm{rng}(S)})(B_{E(x)} S B_v \text{ implies } B_x R B_{F(v)})$$

then having two transitive models $M \subseteq N$ of set theory, such that E and F are absolute according to M and N, the following holds:

(3) if there is in N an S^M-dominating object, then there is an R^M-dominating object, and
(4) if there is in N an R^M-unbounded object, then there is an S^M-unbounded object.

Also "mixed" situations of 3.1.1.1. and the one just above can be considered, i.e. one mapping is absolute, second acts on codes, e.g. fulfilling $\forall x \in C_{\mathrm{dom}(R)}$ $\forall v \in \mathrm{rng}(S)$ $B_{E(x)} S v$ implies $B_x R F(v)$; and all works as before.

5.2. Absoluteness of certain GT-connections.

One of the first who considered absoluteness of certain E- or F-mappings was A.W.Miller (see Lemma 6 and Lemma 7 of [Mi2]), see also [Rep]. The qualitative study of GT-connections was also mentioned in [Fr2]. Results mentioned in this section owes much to a personal discussion with A.Louveau ([Lo]), who helped me overcome some doubtfull steps.

Theorem 5.2.1. *Assume $M \subseteq N$ are transitive models of set theory. Then all mappings $E_2 - E_{12}, F_2 - F_4, F_6 - F_{13}$ (from section 3.2.) are absolute according to M and N.*

Proof. Straightforward, using absoluteness results from [Je].

THEOREM 5.2.2. *Mappings E_1, F_1, F_5 and E_{13} can be defined acting on Borel codes of some substantial part of their domains in such way that for any $M \subseteq N$ models of ZFC are E_1, F_1, F_5 and E_{13} absolute according to M and N.*

PROOF. We use terminology, coding machinery and absoluteness results from [Je].

We deal first with constructions of the Proof of 3.2.1.1. being heavily motivated by [Fr1]
and we show that (1), (2) and (4) are absolute.

(1) is absolute because coding of Borel sets is absolute

(2) is absolute because construction of $V(i,j)$ (see [Fr1]) can be carried out as follows. Start with a topology base $\{U_n : n \in \omega\}$ on $U_i^{\mathcal{I}}$ which is closed under finite unions and put

$$A_k = \left\{ n > k : U_n \cap \bigcap_{m \in I} U_m \neq \emptyset \text{ where } I \subseteq k+1 \text{ and } \bigcap_{m \in I} U_m \neq \emptyset \right\}.$$

The construction of A_k (characteristic function of A_k is a real) is absolute because being empty is an absolute property for Borel sets. And also the sequence $\{A_k : k \in \omega\}$ is absolute as there is an absolute ordering of $\omega \times \omega$. So for fixed i, the countable set of all sequences $\{m_r\}_{r=0}^{(i+1)^2}$ of natural numbers such that $m_{r+1} \in A_{m_r}$ is same in all models and then just put

$$\mathcal{V}_i = \left\{ \bigcup_{r=0}^{(i+1)^2} U_{m_r} : m_0 \text{ arbitrary } \textit{and } m_{r+1} \in A_{m_r} \right\}.$$

The enumeration of $\mathcal{V}_i$ we obtain using (absolute) lexicographical ordering of ${}^{(i+1)^2}\omega$.

(4) is absolute, just take an absolute bijection of $\omega \times \omega$ and ω and construct $G(i,j)$ as the very open basic set with support the image of the set $\{(i,j) \in \omega^2 : j \cdot i \leq k < (j+1) \cdot i\}$.

In the second part of the proof we show that (3) and (5) of 3.2.1.1. and f_X of 3.2.3.5. can be defined on codes.

The absolute code for the operation (3) can be found on $F_\sigma \cap \mathbb{K}$, having a code for $X \in F_\sigma \cap \mathbb{K}$ we can decode nowhere dense sets L_n such that $X = \bigcup L_n$, after this finding a code for the sequence $\left\{ \bigcup_{n=0}^{i} L_n : i \in \omega \right\}$ is primitive recursive.

The absolute code for the operation (5) can be found on $G_\delta \cap \mathbb{L}$, having $Y = \bigcap K_i$ we take first i such that $\mu([0,1] \setminus K_i) < \frac{1}{2}$ and put $B_Y = \{n : \mu(U_n \setminus K_i) = 0\}$, having a recursive enumeration of the base U_n this is an absolute construction and put $K_Y = ([0,1] \setminus K_i) \setminus \bigcup\{U_n : n \in B_Y\}$, it is closed (compact) and the straigtforward code is not worser than δ_3^0.

The last construction which can cause doubts is in 3.2.3.5., where for $X \in G_\delta \cap \mathbb{L}$ we look for an $f_X \in {}^\omega 2$ such that

$$X = \bigcap_n \bigcup \{U_m : m \geq n \textit{and } f_X(m) = 1\}.$$

But from the code of $X = \bigcap L_n$ we can extract (by a definable induction) a sequence $k_{i+1} = \min\{j : j > k_i and\ \mu L_j < \frac{1}{2^{i+1}}\}$ and as $X = \bigcap_n \bigcup_{m=n}^{\infty} L_{k_m}$ we are done.

5.3. PROPERTIES OF SOME FORCING EXTENSIONS.

In this section to ilustrate possible applications of 5.2. we just note a simple application of 5.1. to Cohen and random generic extensions and mention some problems.

5.3.1. Cohen extension. Observe that from the fact that in a simple Cohen extension $N_1 = M[c]$ there is an $(\in_{\mathcal{I}\times\mathbb{K}})^M$-unbounded object (the very Cohen real) and that there are no $(\in_{\mathcal{I}\times\mathbb{K}})^M$-dominating objects then the same holds for $\not\ni_{\mathbb{L}\times\mathcal{I}}$, because of $\in_{\mathcal{I}\times\mathbb{K}} \longrightarrow \not\ni_{\mathbb{L}\times\mathcal{I}}$ (Theorem 3.2.1.2.) and so by 5.1.6. in N_1 there is no $(\in_{\mathcal{I}\times\mathbb{L}})^M$-unbounded object (i.e. no random real over M) but there is an $(\in_{\mathcal{I}\times\mathbb{L}})^M$-dominating object.

5.3.2. Random extension. Analogously for $N_2 = M[r]$ a random extension, from the fact that there is an $(\in_{\mathcal{I}\times\mathbb{L}})^M$-unbounded (the very random real) and no $(\in_{\mathcal{I}\times\mathbb{L}})^M$- dominating object we can conclude that in N_2 there is an $(\in_{\mathcal{I}\times\mathbb{K}})^M$-dominating but no $(\in_{\mathcal{I}\times\mathbb{K}})^M$-unbounded object.

5.3.3. Absoluteness of E- and F-mappings with preservation properties of 5.1.4. and 5.1.5. enables us to compare the strength of some forcing results and find "weakest" forcing doing the job. E.g. For any R in our considerations (which does not generate an Boolean algebra) find a forcing notion adding an R^M-unbounded object such that it does not add any R^M-dominating object and which does not do the same for any simpler S (e.g. $R = \leq^*_{({}^\omega\omega)^2}$ and $S = \in_{\mathcal{I}\times\mathbb{K}}$).

6. Discussion and problems.

6.1. CONVERGENCE VERSUS DIVERGENCE.

The three top relations $\precsim^*_{(\ell^1_+)^2}$, $\supseteq^*_{([\omega]^\omega)^2}$ and $\geq^*_{(\ell^\infty\setminus\ell^1)^2}$ are not directed and the inverse relations are partial orders which generate uniquely Boolean algebras. That is (after a canonical factorization making the orders separative) there are uniquely determined complete Boolean algebras B_K, B_N and B_D such that ℓ^1_+ is dense in B_K, $[\omega]^\omega$ is dense in B_N and $\ell^\infty \setminus \ell^1$ is dense in B_D. In [Vo4] it is proved that

if $\mathfrak{p} = \mathrm{cf}\, 2^{\aleph_0}$ then $B_N \cong B_D$

and

if $\aleph_1 = \mathrm{cf}\, 2^{\aleph_0}$ then $B_N \cong B_K$.

This suggests to formulate the following problem

PROBLEM.

Are $\precsim^*_{(\ell^1_+)^2}$, $\supseteq^*_{([\omega]^\omega)^2}$ and $\geq^*_{(\ell^\infty\setminus\ell^1)^2}$ equivalent according to GT-connections (in ZFC)?

6.2. REAL VALUED SEQUENCES VERSUS TWO VALUED.

In [Vo1] it is proved that

$$\mathfrak{s} = \mathfrak{d}(\chi\mathrm{AOS}_{[\omega]^\omega\times{}^\omega 2}) = \mathfrak{s}_\sigma = \mathfrak{d}(\chi\mathrm{AOS}_{[\omega]^\omega\times\ell^\infty})$$

and it is open whether $\mathfrak{r} = \mathfrak{r}_\sigma$ ([Vol], see Problem 336 of [Va]) . A stronger version of this is the following

PROBLEM.

Is $\chi\text{AOS}_{[\omega]^\omega \times \omega 2}$ simpler than $\chi\text{AOS}_{[\omega]^\omega \times \ell^\infty}$?
Are there any indications that this is possible for RχAOS ?

References

[Ba] T.Bartoszyński, *Additivity of measure implies additivity of category*, Trans. Amer. Math. Soc. **281** (1984), 209–213.

[BaJuSh] T.Bartoszyński, H.Judah, S.Shelah, *The Cichoń diagram*, MSRI 00626-90, November 1989.

[BevD] A.Bešlagič, E.K.van Douwen, *Spaces of subuniform ultrafilters in spaces of uniform ultrafilters*, Preprint.

[BuRecRep] L.Bukovský, I.Reclaw, M.Repický, *Spaces not distinguishing pointwise and quasinormal convergence of real functions*, To appear in Topology Appl..

[Co] R.S.Cooke, *Infinite matrices and sequence spaces*, MacMillan Co., London, 1950.

[vD] E.K.van Douwen, *The integers and topology*, in Handbook of set-theoretic topology, K.Kunen and J.E.Vaughan eds., North-Holland, Amsterdam, 1984, pp. 111–167.

[Fi] G.M.Fichtenholz, *Course of differential and integral calculus*, Russian, vol. **2**, Nauka, Moscow, 1969.

[Fr1] D.H.Fremlin, *Cichoń's diagram*, Sém. d'Initiation á l'Analyse, G. Choquet , M. Rogalski and J. Saint Raymond, Publ. Math. Univ. Pierre et Marie Curie 66, Paris **23** (1983/84), 5.01–5.13.

[Fr2] D.H.Fremlin, *The partially ordered sets of measure theory and Tukey's ordering*, To appear in Note di Matematica.

[Fr3] D.H.Fremlin, *On the relationship between ℓ^1 and $\mathfrak{Z}$.*, Note of 22.8.1991.

[HeHu] H.Herlich, M.Hušek, *Galois connections*, To appear in J.Pure Appl.Algebra.

[Je] T.Jech, *Set theory*, Academic press, New York, 1978.

[JuSh] H.Judah, S.Shelah, *The Kunen-Miller chart*, MSRI 00526-90, November 1989 (November 1989).

[Lo] A.Louveau, Personal discussion, November 1991.

[ML] S.Mac Lane, *Kategorien*, Hochschultext, Springer Verlag, Berlin, 1972.

[Mi1] A.W.Miller, *Some properties of measure and category*, Trans.Amer.Math.Soc. **266** (1981), 93–114.

[Mi2] A.W.Miller, *Additivity of measure implies dominating reals*, Proc. Amer. Math. Soc. **91** (1984), 111–117.

[Mi3] A.W.Miller, *Lectures 1990.*

[Pa] J.Pawlikowski, *Why Solovay real produces Cohen real*, J.Symb.Logic.

[RaSt] J.Raisonier, J.Stern, *The strength of measurability hypothesis*, Israel.J.Math. **50** (1985), 337- 349.

[Rep] M.Repický, *Properties of forcing preserved by finite support iterations*, Comment. Math. Univ. Carolinae **32** (1991), 95–103.

[Ro1] F.Rothberger, *Eine Aequivalenz zwischen der Kontinuumhypothese und der Existenz der Lusinschen und Sierpińskischen Mengen*, Fund.Math **30** (1938), 215–217.

[Ro2] F.Rothberger, *Sur les familles indénombrales de suites de nombres natureles et les problemés concernant la proprieté C*, Proc.Cambridge Phil.Soc. **37** (1941), 109–126.

[St] J.Steprans, *Some cardinal invariants of the continuum related to spliting finite sets*, Preliminary version 1990.

[Tr] J.Truss, *Sets having caliber $\aleph_1$*, in Logic Colloquium' 76, R. O. Gandy and J. M. E. Hyland eds., North-Holland, Amsterdam, 1977, pp. 595 – 612.

[Tu] J.W.Tukey, *Convergence and uniformity in topology*, Princeton University Press, Princeton, Ann. Math. Studies **2** (1940).

[Va] J.E.Vaughan, *Small uncountable cardinals in topology*, in Open problems in topology, J.van Mill and G.M.Reed eds., North-Holland, Amsterdam, 1990, pp. 195–218.

[Vo1] P.Vojtáš, *Cardinalities of posibly noncentered systems of subsets of ω which reflect to some qualities of ultrafilters, p-points and rapid filters*, preprint, presented to Topological conference Baku 1987, published in Vo1a.

[Vo1a] P.Vojtáš, *Cardinalities of noncentered systems of subsets of ω*, Discrete Mathematics **102** (1992).

[Vo2] P.Vojtáš, *Set-theoretic characteristics of summability of sequences and convergence of series*, Comment. Math. Univ. Carolinae **28** (1987), 173–183.

[Vo3] P.Vojtáš, *More on set-theoretic characteristics of summability of sequences by regular (Toeplitz) matrices*, Comment. Math. Univ. Carolinae **29** (1988), 97–102.

[Vo4] P.Vojtáš, *Boolean isomorphism between partial orderings of convergent and divergent series and infinite subsets of* $\mathbb{N}$, To appear in Proc.Amer.Math. Soc.

[Vo5] P. Vojtáš, *Topological cardinal invariants and the Galois-Tukey category* (1992), Akademie Verlag (to appear).