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TOPOLOGICAL CARDINAL INVARIANTS AND THE GALOIS–TUKEY CATEGORY

PETER VOJTÁŠ[1,2]

ABSTRACT. We deal with some interrelations between algebraic theory of categories and set-theoretic topology and with applications of both to real analysis and forcing constructions. We generalize the notion of a Galois connection between partially ordered sets to the one between arbitrary binary relations. We weaken the condition of being the connection by requiring only the implication $E(x)Sv \implies x\mathcal{R}F(v)$ and omiting the monotonicity assumption. In the category consisting of binary relations as objects with the above couple (E, F) considered as a morphism from $\mathcal{R}$ to $\mathcal{S}$ we characterize monos, epis, products and coproducts. We show how these notions are connected with inequalities between cardinal characteristics of relations. Our approach covers in a unified way the study of set-theoretic cardinal invariants of measure, category and other structures of real analysis and topology. Our approach also enables us to introduce generic objects in forcing extensions of models of set theory in a simple way.

1. Introduction and motivation.

Our paper deals with some interrelations between algebraic theory of categories and set-theoretic topology and with applications of both to real analysis and forcing.

Our research is motivated by the study of cardinal characteristics of measure and/or topological structures connected with the real line. The (formerly) pure set-theoretic aspect (dealing with cardinal numbers laying between $\aleph_1$ and $2^{\aleph_0}$), after the work of J.Cichoń, J.Pawlikowski and D.H.Fremlin (see [F]), turned out to have also order-theoretic aspects. Our paper generalizes their approach from partial orders to arbitrary binary relations. We ilustrate both steps (the one of [F] and ours) on two examples. Our first example is the result of T.Bartoszynski ([B]) and J.Raisonnier and J.Stern ([RS]) as reproved in [F] and the second example is the result of F.Rothberger ([Ro]) we reprove using generalized Galois-Tukey connections.

Example 1. Let $\mathbb{L}$ denotes the ideal of Lebesgue measure zero sets and $\mathbb{K}$ the ideal of sets of first category of reals. There are mappings $E : \mathbb{K} \longrightarrow \mathbb{L}$ and $F : \mathbb{L} \longrightarrow \mathbb{K}$ such that $E(A) \subseteq B$ implies $A \subseteq F(B)$, this is (because of the fact that just one implication is fulfilled and E and F are not required to be monotone) a weaker form of a Galois

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[2] Mathematical Institute, Slovak Academy of Sciences, Jesenná 5, CS-041 54 Košice, Czechoslovakia, e-mail: vojtas at csearn.bitnet

connection between partially ordered sets $(\mathbb{K}, \subseteq)$ and $(\mathbb{L}, \subseteq)$ (see [GHKLMS] and [HH]); at the same time E is a Tukey function ([Tu], see [To]). As the consequence we have that the additivity of Lebesgue measure implies the additivity of (topological) Category on reals.

Example 2. In Corolary 8 we show that the proof of Rothberger's result stating that $\text{cov}(\mathbb{L}) \leq \text{non}(\mathbb{K})$ and $\text{non}(\mathbb{L}) \geq \text{cov}(\mathbb{K})$ can be generalized to the assertion that there is a generalized Galois-Tukey connection from $\in \cap([0,1] \times \mathbb{K})$ to $\not\ni \cap(\mathbb{L} \times [0,1])$.

Concerning the algebraic theory of categories we refer to [HS], [ML] and/or [P]; concerning cardinal characteristics we refer to [Va].

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2. Cardinal characteristics of and generalized Galois-Tukey connections between binary relations.

Let (x, y) denote the ordered pair of x and y. A binary relation $\mathcal{R}$ is a set (or also a proper class) of ordered pairs. The domain $\text{dom}(\mathcal{R}) = \{x : (\exists y)((x,y) \in \mathcal{R})\}$ and the range $\text{rng}(\mathcal{R}) = \{y : (\exists x)((x,y) \in \mathcal{R})\}$. The fact $(x,y) \in \mathcal{R}$ will be often written as $x\mathcal{R}y$. The complement (or negation) of $\mathcal{R}$ is denoted by $\neg\mathcal{R} = \{(x,y) : x \in \text{dom}(\mathcal{R})\ \&\ y \in \text{rng}(\mathcal{R})\ \&\ (x,y) \notin \mathcal{R}\}$. The inverse $\mathcal{R}^{-1} = \{(y,x) : (x,y) \in \mathcal{R}\}$. Observe that in general $\neg\neg\mathcal{R} = \mathcal{R}$ does not necessarily hold. For a set X the cardinality of X is denoted by $|X|$.

Definition 3. A set $B \subseteq \text{dom}(\mathcal{R})$ is said to be $\mathcal{R}$-unbounded if $(\forall y \in \text{rng}(\mathcal{R}))\ (\exists x \in B)$ $((x,y) \notin \mathcal{R})$. A set $D \subseteq \text{rng}(\mathcal{R})$ is said to be $\mathcal{R}$-dominating if $(\forall x \in \text{dom}(\mathcal{R}))\ (\exists y \in D)$ $((x,y) \in \mathcal{R})$.

Observe that for a binary relation $\mathcal{R}$ if B_1 is $\mathcal{R}$-unbounded and $B_1 \subseteq B_2 \subseteq \text{dom}(\mathcal{R})$ then B_2 is $\mathcal{R}$-unbounded, too, and if D_1 is $\mathcal{R}$-dominating and $D_1 \subseteq D_2 \subseteq \text{rng}(\mathcal{R})$ then D_2 is $\mathcal{R}$-dominating, too. Moreover $\text{rng}(\mathcal{R})$ is always $\mathcal{R}$-dominating and $\text{dom}(\mathcal{R})$ is $\mathcal{R}$-unbounded if $\text{rng}(\mathcal{R}) = \text{rng}(\neg\mathcal{R})$. For $\text{dom}(\neg\mathcal{R})$ to be $\neg\mathcal{R}$-unbounded it is sufficient (but not necessary) that $\text{dom}(\mathcal{R}) = \text{dom}(\neg\mathcal{R})$. Further if $B \subseteq \text{dom}(\mathcal{R}) = \text{rng}(\mathcal{R}^{-1}) \supseteq \text{rng}(\neg\mathcal{R}^{-1})$ is $\mathcal{R}$-unbounded, then $B \cap \text{rng}(\neg\mathcal{R}^{-1})$ is $\neg\mathcal{R}^{-1}$-dominating (as $\forall y \in \text{dom}(\neg\mathcal{R}^{-1}) \subseteq \text{dom}(\mathcal{R}^{-1}) = \text{rng}(\mathcal{R})$, the corresponding $x \in B$ with $(x,y) \notin \mathcal{R}$ fulfills $x \in \text{rng}(\neg\mathcal{R}^{-1})$ and $(y,x) \in \neg\mathcal{R}^{-1}$). Similarly, if $D \subseteq \text{rng}(\mathcal{R}) = \text{dom}(\mathcal{R}^{-1}) \supseteq \text{dom}(\neg\mathcal{R}^{-1})$ is $\mathcal{R}$-dominating, then $D \cap \text{dom}(\neg\mathcal{R}^{-1})$ is $\neg\mathcal{R}^{-1}$-unbounded.

For further applications (namely of Example 2) we would like to have $\mathcal{R} = \neg\neg\mathcal{R}$ and to avoid in Definition 5 undefined cases for $\mathcal{R}$,$\mathcal{R}^{-1}$,$\neg\mathcal{R}$ and $\neg\mathcal{R}^{-1}$ we make the following

Restriction 4. In what follows all relations considered are assumed to fulfill
$\text{dom}(\mathcal{R}) = \text{dom}(\neg\mathcal{R})$ and $\text{rng}(\mathcal{R}) = \text{rng}(\neg\mathcal{R})$.

Definition 5. Assume $\mathcal{R}$ is a binary relation (with the above restriction). We define $\mathfrak{b}(\mathcal{R}) = \min\{|B| : B \text{ is an } \mathcal{R}\text{-unbounded set}\}$ and $\mathfrak{d}(\mathcal{R}) = \min\{|D| : D \text{ is an } \mathcal{R}\text{-dominating set }\}$.

Under our restriction, from the discussion following Definition 3 we get

Observation 6. Assume $\mathcal{R}$ is a binary relation. All of the following numbers are defined and $\mathfrak{b}(\mathcal{R}) = \mathfrak{d}(\neg\mathcal{R}^{-1}) \geq 2$, $\mathfrak{d}(\mathcal{R}) = \mathfrak{b}(\neg\mathcal{R}^{-1}) \geq 2$, $\mathfrak{b}(\mathcal{R}^{-1}) = \mathfrak{d}(\neg\mathcal{R}) \geq 2$ and $\mathfrak{d}(\mathcal{R}^{-1}) = \mathfrak{b}(\neg\mathcal{R}) \geq 2$.

The next definition appears in [Vo] and paraphrases the corresponding definition in [HH], [GHKLMS] and [Tu], see[To] and also [F].

Definition 7. Assume $\mathcal{R}$ and $\mathcal{S}$ are binary relations. An ordered pair of functions (E, F) is said to be a generalized Galois-Tukey connection from $\mathcal{R}$ to $\mathcal{S}$ if the following holds:

(a) $E : \text{dom}(\mathcal{R}) \longrightarrow \text{dom}(\mathcal{S})$,
(b) $F : \text{rng}(\mathcal{S}) \longrightarrow \text{rng}(\mathcal{R})$,
(c) $(\forall x \in \text{dom}(\mathcal{R}))(\forall v \in \text{rng}(\mathcal{S}))E(x)\mathcal{S}v$ implies $x\mathcal{R}F(v)$.

Observe that under our restriction (c) is always fulfilled nontrivially, i.e. there is always some v, $\mathcal{S}$-dominating given $E(x)$ and moreover for an $\mathcal{R}$-unbounded set B, $E(B)$ is $\mathcal{S}$-unbounded, and similarly, if D is $\mathcal{S}$-dominating, then $F(D)$ is $\mathcal{R}$-dominating. As $\mathrm{dom}(\mathcal{R}) = \mathrm{rng}(\neg\mathcal{R}^{-1})$ and $\mathrm{rng}(\mathcal{R}) = \mathrm{dom}(\neg\mathcal{R}^{-1})$ (similarly for $\mathcal{S}$) and as implication (c) is equivalent to $F(v)\neg\mathcal{R}^{-1}x$ implies $v\neg\mathcal{S}^{-1}E(x)$ we have that the ordered pair (F, E) is a generalized Galois-Tukey connection from $\neg\mathcal{S}^{-1}$ to $\neg\mathcal{R}^{-1}$. This discussion provides a proof of the following

Observation 8. Assume there is a generalized Galois-Tukey connection from $\mathcal{R}$ to $\mathcal{S}$, then $\mathfrak{d}(\neg\mathcal{S}^{-1}) = \mathfrak{b}(\mathcal{S}) \leq \mathfrak{b}(\mathcal{R}) = \mathfrak{d}(\neg\mathcal{R}^{-1})$ and $\mathfrak{b}(\neg\mathcal{S}^{-1}) = \mathfrak{d}(\mathcal{S}) \geq \mathfrak{d}(\mathcal{R}) = \mathfrak{b}(\neg\mathcal{R}^{-1})$.

This approach unifies a lot of definitions of cardinal characteristics (at least those defined as minimum of cardinalities of sets with a given property) studied in topology, boolean algebras and real analysis (see [Va], [Ju] and [Vo]). Namely $\mathrm{non}(\mathbb{L}) = \mathfrak{b}(\in \cap([0,1] \times \mathbb{L}))$ and $\mathrm{cov}(\mathbb{L}) = \mathfrak{d}(\in \cap([0,1] \times \mathbb{L}))$, similarly for $\mathbb{K}$. Using Observation 8 we can finish Example 2 by the following

Corollary 9. There is a generalized Galois-Tukey connection from $\mathcal{R} = \in \cap([0,1] \times \mathbb{K})$ to $\mathcal{S} = \not\ni \cap(\mathbb{L} \times [0,1])$ and consequently the result of [Ro] follows, i.e. $\mathrm{cov}(\mathbb{L}) \leq \mathrm{non}(\mathbb{K})$ and $\mathrm{non}(\mathbb{L}) \geq \mathrm{cov}(\mathbb{K})$.

Proof. Take $A \in \mathbb{L}$ such that A is comeager and let $B = \mathcal{R} \setminus A$. Let $E : [0,1] \longrightarrow \mathbb{L}$ and $F : [0,1] \longrightarrow \mathbb{K}$ be defined by putting $E(x) = x + A$ and $F(y) = y - B$ (where e.g. $x + A = \{x + y : y \in A\}$) and note that $x + A \not\ni y$ implies $x \in y - B$. So by Observation 8 we have $\mathrm{cov}(\mathbb{L}) = \mathfrak{d}(\neg\mathcal{S}^{-1}) = \mathfrak{b}(\mathcal{S}) \leq \mathfrak{b}(\mathcal{R}) = \mathrm{non}(\mathbb{K})$ and $\mathrm{non}(\mathbb{L}) = \mathfrak{b}(\neg\mathcal{S}^{-1}) = \mathfrak{d}(\mathcal{S}) \geq \mathfrak{d}(\mathcal{R}) = \mathrm{cov}(\mathbb{K})$.

3. The Galois-Tukey category $\mathbb{GT}$.

Classical Galois connections can be considered as adjoint functors between the partial orders seen as categories. Although it is not clear whether our generalized connections (we omit now the problem with just $\Rightarrow$ instead of $\Leftrightarrow$ and that the monotonicity requirement is also omited) could be seen as a Galois-connections of the third kind (see [HH]), where $\mathcal{R}$ and $\mathcal{S}$ are considered as concrete categories above some category, we are not going this way. Our intended interpretation and/or the heuristics of our applications leads us "one level higher" - roughly speaking, to functorial categories, i.e. where functors (in a class of small categories) become morphisms. There is still posible to work with a certain subcategory (namely of adjoint functors) of the product of the functorial category and its opposite. Having all this in mind and taking a simpler way we give the following

Definition 10. The Galois-Tukey category (denoted $\mathbb{GT}$) consists of binary relations as objects (under our restriction) and a couple $(\mathcal{R}, \mathcal{S}, E, F)$ is a morphism of $\mathbb{GT}$ from $\mathcal{R}$ to $\mathcal{S}$ if (E, F) is a generalizad Galois-Tukey connection from $\mathcal{R}$ to $\mathcal{S}$. The identity on $\mathcal{R}$ is $(\mathcal{R}, \mathcal{R}, id_{\mathrm{dom}(\mathcal{R})}, id_{\mathrm{rng}(\mathcal{R})})$. The composition being defined as follows $(\mathcal{S}, \mathcal{T}, H, K) \circ (\mathcal{R}, \mathcal{S}, E, F) = (\mathcal{R}, \mathcal{T}, H \circ E, F \circ K)$.

The following will help us to construct (co)products and characterize epis and monos.

Observation 11. (1) $\mathbb{F}_1 : \mathbb{GT} \longrightarrow Set \times Set^{op}$, where $\mathbb{F}_1(\mathcal{R}) = (\mathrm{dom}(\mathcal{R}), \mathrm{rng}(\mathcal{R}))$ and $\mathbb{F}_1((\mathcal{R}, \mathcal{S}, E, F)) = ((\mathrm{dom}(\mathcal{R}), E, \mathrm{dom}(\mathcal{S})), (\mathrm{rng}(\mathcal{S}), F, \mathrm{rng}(\mathcal{R})))$ is a (covariant) functor.
(2) $\mathbb{F}_2 : \mathbb{GT} \longrightarrow Card^{op} \times Card$, where $Card$ is the category of cardinal numbers and $\mathfrak{m} \leq \mathfrak{n}$ iff there is exactly one morphism from $\mathfrak{m}$ to $\mathfrak{n}$ and $\mathbb{F}_2(\mathcal{R}) = (\mathfrak{b}(\mathcal{R}), \mathfrak{d}(\mathcal{R}))$ is a (covariant) functor.
(3) $\mathbb{F}_3 : \mathbb{GT} \longrightarrow \mathbb{GT}$ defined on objects by $\mathbb{F}_3(\mathcal{R}) = \neg\mathcal{R}^{-1}$ and on morphisms by $\mathbb{F}_3((\mathcal{R}, \mathcal{S}, E, F)) = ((\neg\mathcal{S}^{-1}, \neg\mathcal{R}^{-1}, F, E))$ is a contravariant functor.

Proof. (1) is clear from Definition 10, (2) is in Observation 8 and (3) is in the discussion after the Definition 7.

Having in mind that Hausdorff spaces are characterized as spaces where epis are exactly continuous mappings onto a dense set, we obtain the following notions (playing an important role in applications).

Definition 12. (1) A binary relation $\mathcal{R}$ is said to be a Hausdorff relation if for all $B \subseteq \mathrm{dom}(\mathcal{R})$ if B is $\mathcal{R}$-bounded then for every $x \in \mathrm{dom}(\mathcal{R})$ the set $B \cup \{x\}$ is still $\mathcal{R}$-bounded. The full subcategory of $\mathbb{GT}$ consisting of Hausdorff relations we denote $\mathbb{GTH}$.
(2) A binary relation $\mathcal{R}$ is said to have the trichotomy property if for all $y_1, y_2 \in \mathrm{rng}(\mathcal{R})$ is either $\mathcal{R}^{-1}(y_1) \subseteq \mathcal{R}^{-1}(y_2)$ or $\mathcal{R}^{-1}(y_2) \subseteq \mathcal{R}^{-1}(y_1)$. The full subcategory of $\mathbb{GTH}$ consisting of relations with the trichotomy property we denote $\mathbb{GTHT}$.

Observation 13. (1) Monos of $\mathbb{GTH}$ are exactly monos of $Set \times Set^{op}$. (2) Epis of $\mathbb{GTHT}$ are exactly epis of $Set \times Set^{op}$.

Proof. Given a mapping J, $x \in \mathrm{dom}(J)$ and $y \in \mathrm{rng}(J)$ we denote $J_{x/y}$ the mapping defined everywhere to be equal to J except for x, where $J_{x/y}(x) = y$. It suffices to notice the following: For $\mathcal{R} \in Obj(\mathbb{GTH})$ and for every $x_1, x_2 \in \mathrm{dom}(\mathcal{R})$ the couple $(\mathcal{R}, \mathcal{R}, (id_{\mathrm{dom}(\mathcal{R})})_{x_1/x_2}, id_{\mathrm{rng}(\mathcal{R})})$ is a morphism and for all $y_1 \in \mathrm{rng}(\mathcal{R})$ there is an $y_2 \in \mathrm{rng}(\mathcal{R})$ such that $y_1 \neq y_2$ and $(\mathcal{R}, \mathcal{R}, id_{\mathrm{dom}(\mathcal{R})}, (id_{\mathrm{rng}(\mathcal{R})})_{y_1/y_2})$ is a morphism. Finaly for $\mathcal{R} \in Obj(\mathbb{GTHT})$ and for all $y_1, y_2 \in \mathrm{rng}(\mathcal{R})$ the $(\mathcal{R}, \mathcal{R}, id_{\mathrm{dom}(\mathcal{R})}, (id_{\mathrm{rng}(\mathcal{R})})_{y_1/y_2})$ is a morphism.

In contrast to the similarity with Hausdorff spaces mentioned above we do not know whether $\mathbb{GTH}$ and $\mathbb{GTHT}$ are maximal categories with the given property.

4. Products and coproducts in $\mathbb{GT}$.

In this section we show that $\mathbb{GT}$ has products and coproducts. The fact is not only interesting from the point of view of theory of categories but has several applications to set-theoretic structures. In what follows $\pi_x : X \times U \longrightarrow X$ denotes the projection to the first coordinate (similarly π^u) and $id_Y \times 0 : Y \longrightarrow Y \times \{0\}$ just denotes identical map of Y onto a disjoint copy of Y (similarly $id_V \times 1$).

Theorem 14. The relation $\mathcal{R} \otimes \mathcal{S} = \{((x,u),(a,\epsilon)) : x\mathcal{R}a \ \& \ \epsilon = 0 \text{ or } u\mathcal{S}a \ \& \ \epsilon = 1\}$ together with the morphisms $(\mathcal{R} \otimes \mathcal{S}, \mathcal{R}, E_1 = \pi_x, F_1 = id_{\mathrm{rng}(\mathcal{R})} \times 0)$ and $(\mathcal{R} \otimes \mathcal{S}, \mathcal{S}, E_2 = \pi^u, F_2 = id_{\mathrm{rng}(\mathcal{S})} \times 1)$ form the product of $\mathcal{R}$ and $\mathcal{S}$ in $\mathbb{GT}$.

Proof. Let $(\mathcal{T}, \mathcal{R}, H_1, K_1)$ and $(\mathcal{T}, \mathcal{S}, H_2, K_2)$ be arbitrary morphisms of $\mathbb{GT}$. Then there is a (unique) morphism $(\mathcal{T}, \mathcal{R} \otimes \mathcal{S}, I, J)$ such that $E_1 \circ I = H_1, J \circ F_1 = K_1$, $E_2 \circ I = H_2$ and $J \circ F_2 = K_2$, namely $I(w) = (H_1(w), H_2(w))$ and $J(a,\epsilon) = K_1(a)$ if $\epsilon = 0$ and $J(a,\epsilon) = K_2(a)$ if $\epsilon = 1$.

Theorem 15. The relation $\mathcal{R} \oplus \mathcal{S} = \{((a,\epsilon),(y,v)) : a\mathcal{R}y \ \& \ \epsilon = 0 \text{ or } a\mathcal{S}v \ \& \ \epsilon = 1\}$ together with morphisms $(\mathcal{R}, \mathcal{R} \oplus \mathcal{S}, id_{\mathrm{dom}(\mathcal{R})} \times 0, \pi_y)$ and $(\mathcal{S}, \mathcal{R} \oplus \mathcal{S}, id_{\mathrm{dom}(\mathcal{S})} \times 1, \pi^v)$ form the coproduct of $\mathcal{R}$ and $\mathcal{S}$ in $\mathbb{GT}$.

Proof. Similar as above.

Arbitrary products and coproducts can be constructed similarly.

Corollary 16. (1) The functor $\mathbb{F}_1$ from Observation 11 preserves products and coproducts. (2) The functor $\mathbb{F}_3$ from Observation 11 preserves products and coproducts, as undestood for a contravariant functor.

Proof. (1) Constructions of Theorem 14 and 15 are transformed by $\mathbb{F}_1$ to (co)products in $Set \times Set^{op}$. (2) Just notice that $\mathcal{R} \oplus \mathcal{S} = \neg(\neg\mathcal{R}^{-1} \otimes \neg\mathcal{S}^{-1})^{-1}$ and vice versa.

To show that $\mathbb{F}_2$ from Observation 11 also preserves products and coproducts we hawe to count cardinal characteristics of $\mathcal{R} \otimes \mathcal{S}$ and $\mathcal{R} \oplus \mathcal{S}$.

Theorem 17. $\mathfrak{b}(\mathcal{R} \otimes \mathcal{S}) = \max(\mathfrak{b}(\mathcal{R}), \mathfrak{b}(\mathcal{S}))$ and $\mathfrak{d}(\mathcal{R} \otimes \mathcal{S}) = \min(\mathfrak{d}(\mathcal{R}), \mathfrak{d}(\mathcal{S}))$ and $\mathfrak{b}(\mathcal{R} \oplus \mathcal{S}) = \min(\mathfrak{b}(\mathcal{R}), \mathfrak{b}(\mathcal{S}))$ and $\mathfrak{d}(\mathcal{R} \oplus \mathcal{S}) = \max(\mathfrak{d}(\mathcal{R}), \mathfrak{d}(\mathcal{S}))$

Proof. Just notice that if B_1 is $\mathcal{R}$ -unbounded and B_2 is $\mathcal{S}$-unbounded then $B_1 \times B_2$ is $\mathcal{R} \otimes \mathcal{S}$-unbounded. Moreover if D_1 is $\mathcal{R}$-dominating and D_2 is $\mathcal{S}$-dominating then both $D_1 \times \{0\}$ and $D_2 \times \{1\}$ are $\mathcal{R} \otimes \mathcal{S}$-dominating. The rest follows from Observation 8 and from (2) in Corollary 16.

Corollary 18. The functor $\mathbb{F}_2$ from Observation 11 preserves products and coproducts.

Proof. Easy, just note that max and min represent coproduct and product in the category *Card*.

5. Generic objects according to binary relations and direct sums and products of forcing notions.

Consider two models of set theory $M \subseteq N$. We restrict ourselves to structures which are definable, i.e. their construction can be carried out in both M and N. So, for instance we have $(Reals)^M$ and $(Reals)^N$ as constructed in M and N, respectively. Roughly speaking, we can consider $(Reals)^M$ as a subfield of $(Reals)^N$. Similarly, (more difficult) we can consider the ideal $\mathbb{K}$ in M and N and moreover $(\mathbb{K}^M)^N$ consisting of sets defined as in M but carried out in N (e.g. starting with open intervals with endpoints a, b from M but consisting of all reals in N between a and b, then forming open dense sets in the same way as in M just from this intervals interpreted in N and finaly, taking the same intersection of countably many open dense sets as in M just using open dense sets already interpreted in N). More exact description of and unexplained notions are in [Je]. Having in mind this "rough example" we define

Definition 19. Assume $M \subseteq N$ are two models of set theory. Then an $x \in \mathrm{dom}(\mathcal{R})^N$ is said to be an $\mathcal{R}^M$-unbounded object if for all $y \in \mathrm{rng}(\mathcal{R})^M$ it is not the case that $x\mathcal{R}^N y^N$; and an $y \in \mathrm{rng}(\mathcal{R})^N$ is said to be an $\mathcal{R}^M$-dominating object if for all $x \in \mathrm{dom}(\mathcal{R})^M$ we have $x^N \mathcal{R}^N y$.

Observation 20. Asume $M \subseteq N$ are two models of set theory, then: (1) In N, Cohen reals over M are exactly all $\in \cap([0,1] \times \mathbb{K})^M$-unbounded objects; (2) In N, Random reals over M are exactly all $\in \cap([0,1] \times \mathbb{L})^M$-unbounded objects.

Proof. See characterizations of Cohen and/or Random reals in [Je].

Theorem 21. Assume B_1 and B_2 are complete Boolean algebras and B_1 adds $\mathcal{R}^M$-unbounded objects and B_2 adds $\mathcal{S}^M$-unbounded objects. Then $B_1 \oplus B_2$ adds $(\mathcal{R} \oplus \mathcal{S})^M$-unbounded objects and $B_1 \otimes B_2$ adds $(\mathcal{R} \otimes \mathcal{S})^M$-unbounded objects.

Proof. Similar as the proof of the Theorem 17. Assume x is an $\mathcal{R}^M$-unbounded object, y is an $\mathcal{S}^M$-unbounded object. Then (x, y) is an $\mathcal{R} \otimes \mathcal{S}$-unbounded object and both $(x, 0)$ and $(y, 1)$ are $\mathcal{R} \oplus \mathcal{S}$-unbounded objects.

Several natural questions arise: (1) study certain subcategories of $\mathbb{GT}$, e.g. consisting of relations of cardinality less than or equal to continuum or of Borel relations on reals; (2) study certain factorizations of $\mathbb{GT}$, e.g. factorize all morphisms $(\mathcal{R}, \mathcal{S}, E, F_1)$ and $(\mathcal{R}, \mathcal{S}, E, F_2)$, or factorize all objects, which are isomorphic; (3) Factorize the whole $Hom(\mathcal{R}, \mathcal{S})$ to a single morphism. Then the resulting category is a category of a pseudo partial order. Investigate categorical and/or order-theoretic properties of this structure. Or even more, take just definable relations; (4) carry out other forcing constructions, e.g. iteration, by categorical construction or reformulate Observation 20 and Theorem 21 for $\mathcal{R}^M$-dominating objects and apply it. All this and something more will be studied in a forthcomming paper.

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