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Cardinalities of noncentered systems of subsets of ω

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Abstract

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We introduce a couple of new cardinal characteristics of ω^* which are equal to the minimal size of a system of infinite subsets of ω satisfying some properties which were till now considered as a quality of ultrafilters, p-points and rapid filters respectively. From this systems we do not require centeredness and hence they always exist. We give some estimations and equivalent reformulations and state problems concerning these new cardinal characteristics.

In the context of summation methods it turned out to be useful to consider some new cardinal characteristics of ω^* (see [7, 11, 12]). In the present paper we study these cardinal characteristics in their own right.

Our notation is the standard one used in set-theoretic topology (see e.g. [4]): ω denotes the set of natural numbers. $\omega^* = \beta\omega - \omega$ denotes the remainder of the Čech–Stone compactification of the natural numbers equipped with the discrete topology (and we keep in mind the Stone duality), $[X]^\omega$ is the system of all countably infinite subsets of X, ℓ^∞ is the system of all bounded sequences of real numbers, $A \subseteq^* B$ holds if $A - B$ is finite, ${}^XY$ denotes the set of all mappings from X into Y.

We say that a family $\mathcal{S} \subseteq [\omega]^\omega$ is *splitting* (see e.g. [4]) if for every $X \in [\omega]^\omega$ there is an $S \in \mathcal{S}$ such that $|S \cap X| = |X - S| = \aleph_0$.

We say that a family $\mathcal{R} \subseteq [\omega]^\omega$ is *refining* if for every $X \in [\omega]^\omega$ there is an $R \in \mathcal{R}$ such that either $R \subseteq^* X$ or $R \subseteq^* \omega - X$.

We say that a family $\mathcal{C} \subseteq \ell^\infty$ is *chaotic* in ℓ^∞ if for every $X \in [\omega]^\omega$ there is a $c \in \mathcal{C}$ such that $\lim\{c(n): n \in X\}$ does not exist.

We say that a family $\mathcal{A} \subseteq [\omega]^\omega$ is *attractive* for ℓ^∞ if for every $c \in \ell^\infty$ there is an $X \in \mathcal{A}$ such that $\lim\{c(n): n \in X\}$ does exist. We denote the corresponding cardinal characteristics by

$$\mathfrak{s} = \min\{|\mathcal{S}|: \mathcal{S} \text{ is a splitting family}\},$$

$$\mathfrak{r} = \min\{|\mathcal{R}|: \mathcal{R} \text{ is a refining family}\},$$

$$\mathfrak{s}_\sigma = \min\{|\mathcal{C}|: \mathcal{C} \subseteq \ell^\infty \text{ is a chaotic family}\},$$

$$\mathfrak{r}_\sigma = \min\{|\mathcal{A}|: \mathcal{A} \text{ is an attractive family for } \ell^\infty\}.$$

The notation of $\mathfrak{r}$ (r refers to refining) and $\mathfrak{s}$ (s refers to splitting) is mnemonic. The notation of $\mathfrak{r}_\sigma$ and $\mathfrak{s}_\sigma$ is mnemonic too, see Theorem B ($\mathfrak{r}_\sigma$ refers to σ-refining and $\mathfrak{s}_\sigma$ refers to σ-splitting).

Recall that

$$\mathfrak{u} = \min\{|\mathcal{B}|: \mathcal{B} \text{ is a neighborhood base of a point in } \omega^*\};$$

$$\mathfrak{u}_p = \begin{cases} \min\{|\mathcal{B}|: \mathcal{B} \text{ is a neighborhood base of a } p\text{-point in } \omega^*\} \\ \quad \text{if there are } p\text{-points,} \\ 2^\omega \quad \text{otherwise,} \end{cases}$$

$\mathbb{L} \subseteq \mathcal{P}(\langle 0, 1\rangle)$ is the ideal of Lebesgue measure zero sets; $\mathbb{K} \subset \mathcal{P}(\langle 0, 1\rangle)$ is the ideal of meagre sets; $\mathrm{Cov}(\mathbb{L} \cap \mathbb{K}) = \min\{|\mathcal{X}|: \mathcal{X} \subseteq \mathbb{L} \cap \mathbb{K} \text{ and } \bigcup \mathcal{X} = \langle 0, 1\rangle\}$; $\mathrm{Non}(\mathbb{L} \cap \mathbb{K}) = \min\{|X|: X \subseteq \langle 0, 1\rangle \text{ and } X \notin \mathbb{L} \cap \mathbb{K}\}$.

Theorem A. (i) *Every ultrafilter base is refining and* $\mathfrak{r} \leqslant \mathfrak{u}$.

(ii) *Every p-point base is attractive for* ℓ^∞ *and* $\mathfrak{r}_\sigma \leqslant \mathfrak{u}_p$.

(iii) $\mathfrak{s}_\sigma \leqslant \mathfrak{s}$ *and* $\mathfrak{r} \leqslant \mathfrak{r}_\sigma$.

(iv) $\mathfrak{r} \geqslant \mathrm{Cov}(\mathbb{L} \cap \mathbb{K})$ *and* $\mathfrak{s} \leqslant \mathrm{Non}(\mathbb{L} \cap \mathbb{K})$.

Proof. (i) and (ii) are easy.

(iii) If $\mathcal{S}$ is a splitting family then the system of characteristic functions $\{\chi_S: S \in \mathcal{S}\}$ is a chaotic family, which gives $\mathfrak{s}_\sigma \leqslant \mathfrak{s}$. Moreover, if $\mathcal{A}$ is an attractive family for ℓ^∞ then it is also attractive for ${}^\omega 2$, i.e., for every $f \in {}^\omega 2$ there is an $X \in \mathcal{A}$ such that $\lim\{f(n): n \in X\}$ exists; but this is possible if and only if either $X \subseteq^* f^{-1}(0)$ or $X \subseteq^* f^{-1}(1)$ holds true, i.e., $\mathcal{A}$ is a refining family.

(iv) We first follow the proof of Brzuchowski [2]. Assume $\mathcal{R}$ is a refining family and for $R \in \mathcal{R}$ put

$$X(R) = \{f \in {}^\omega 2: R \subseteq^* f^{-1}(0) \text{ or } R \subseteq^* f^{-1}(1)\}.$$

Clearly $X(R) \in \mathbb{L} \cap \mathbb{K}$ and $\bigcup\{X(R): R \in \mathcal{R}\} = {}^\omega 2$, which gives $\mathfrak{r} \geqslant \mathrm{Cov}(\mathbb{L} \cap \mathbb{K})$.

To prove the second assertion, let $\mathcal{F} \subset {}^\omega 2$ be arbitrary with $|\mathcal{F}| < \mathfrak{s}$. Then $\{f^{-1}(0): f \in \mathcal{F}\}$ is not a splitting family and there is an $R \in [\omega]^\omega$ such that

$$(\forall f \in \mathcal{F})(R \subseteq^* f^{-1}(0) \quad \text{or} \quad R \subseteq^* f^{-1}(1)),$$

i.e., $\mathcal{F} \subseteq X(R)$, i.e., $\mathcal{F} \in \mathbb{L} \cap \mathbb{K}$. $\square$

In the proof of (iii) of Theorem A it was implicitly shown that splitting families correspond to chaotic families in ${}^\omega 2$ and refining families correspond to attractive families for ${}^{\omega}2$. The question is whether also $\mathfrak{r}_\sigma$ and $\mathfrak{s}_\sigma$ can be reformulated using only properties of subsets of natural numbers.

We say that a family $\mathbb{S} \subseteq [[\omega]^\omega]^\omega$ is *σ-splitting* if for every $X \in [\omega]^\omega$ there is an $\mathcal{S} \in \mathbb{S}$ and $S \in \mathcal{S}$ such that $|S \cap X| = |X - S| = \aleph_0$.

We say that a family $\mathcal{R} \subseteq [\omega]^\omega$ is *σ-refining* if for every $\mathcal{S} \in [[\omega]^\omega]^\omega$ there is an $R \in \mathcal{R}$ such that for every $S \in \mathcal{S}$ either $R \subseteq^* S$ or $R \subseteq^* \omega - S$.

Theorem B. (i) $\mathfrak{s} = \min\{|\mathcal{C}|$: *$\mathcal{C} \subset {}^{\omega}2$ is a chaotic family*$\}$,

(ii) $\mathfrak{r} = \min\{|\mathcal{A}|$: *$\mathcal{A}$ is an attractive family for ${}^{\omega}2$*$\}$,

(iii) $\mathfrak{s}_\sigma = \min\{|\mathbb{S}|$: *$\mathbb{S}$ is a σ-splitting family*$\}$,

(iv) $\mathfrak{r}_\sigma = \min\{|\mathcal{R}|$: *$\mathcal{R}$ is a σ-refining family*$\}$,

(v) $\mathfrak{s} = \mathfrak{s}_\sigma$.

Proof. (i) and (ii) are obvious.

(iii) and (iv). For $a \in \ell^\infty$ and $q \in Q$ (Q being the set of all rational numbers) put $L(a, q) = \{n: a(n) < q\}$. Observe that if $X \in [\omega]^\omega$ is such that for all $q \in Q$ we have $X \subseteq^* L(a, q)$ or $X \subseteq^* \omega - L(a, q)$ then $\lim\{a(n): n \in X\}$ exists. Now if $\mathcal{C} \subseteq \ell^\infty$ is a chaotic family then for every $a \in \mathcal{C}$ put $\mathcal{S}(a) = \{L(a, q): q \in Q\}$. Then $\mathbb{S}_{\mathcal{C}} = \{\mathcal{S}(a) \cap [\omega]^\omega: a \in \mathcal{C}\}$ is a σ-splitting family. This gives $\geqslant$ in (iii). Moreover if $\mathcal{R}$ is a σ-refining family then $\mathcal{R}$ is an attractive family for ℓ^∞. This gives $\leqslant$ in (iv).

To prove the remaining inequalities we need the following construction. Take an $\mathcal{S} = \{S(n): n \in \omega\} \subseteq [\omega]^\omega$ and for $n \in \omega$, $s \in {}^{n}2$ put

$$B(s) = S(0)^{s(0)} \cap S(1)^{s(1)} \cap \cdots \cap S(n-1)^{s(n-1)},$$

where $X^0 = X$ and $X^1 = \omega - X$, and define $c_{\mathcal{S}}: \omega \to \langle 0, 1\rangle$ as follows:

$$c_{\mathcal{S}}(n) = \sum_{i \in \omega} 2 \cdot s(i)/3^i \quad s \in {}^{i+1}2,\ n \in B(s).$$

Observe that if $X \in [\omega]^\omega$ is such that $\lim\{c_{\mathcal{S}}(n): n \in X\}$ exists then X refines $\mathcal{S}$, i.e., for every $n \in \omega$ either $X \subseteq^* S(n)$ or $X \subseteq^* \omega - S(n)$. Clearly if $\mathbb{S}$ is a σ-splitting family then $\{c_{\mathcal{S}}: \mathcal{S} \in \mathbb{S}\}$ is a chaotic family in ℓ^∞ and if $\mathcal{A}$ is an attractive family for ℓ^∞ then $\mathcal{A}$ is also a σ-refining family.

(v) Using (iii) of this theorem and (iii) of Theorem A it is enough to observe that if $\mathbb{S} \subseteq [[\omega]^\omega]^\omega$ is a σ-splitting family then $\bigcup \mathbb{S} \subseteq [\omega]^\omega$ is a splitting family and $|\mathbb{S}| = |\bigcup \mathbb{S}|$. □

Problems. Is it true (in ZFC) that $\mathfrak{r} = \mathfrak{r}_\sigma$?

Is it consistent with ZFC that $\mathfrak{r} < \mathfrak{u}$?

Is it consistent with ZFC that $\mathfrak{r}_\sigma < \mathfrak{u}_p$?

To motivate the next definition, recall that we obtained $\mathfrak{r}$ and $\mathfrak{r}_\sigma$ by omitting the requirement of centeredness using properties of ultrafilters and p-points.

There is yet another notion studied only in the 'centered version', the notion of a rapid filter (see [9]).

Write $c_0^+ = \{a\colon a$ is a mapping from ω to $<0, +\infty)$ & $\lim a(n) = 0\}$. Recall that

$$\mathfrak{b} = \min\{|\mathscr{B}|\colon \mathscr{B} \subseteq {}^\omega\omega \text{ is unbounded under } <^* \text{ in } {}^\omega\omega\}$$

and

$$\mathfrak{d} = \min\{|\mathscr{D}|\colon \mathscr{D} \subseteq {}^\omega\omega \text{ is dominating under } <^* \text{ in } {}^\omega\omega\},$$

see also [4].

Theorem C. $\mathfrak{d} = \min\{|\mathscr{A}|\colon \mathscr{A} \subseteq [\omega]^\omega$ *and for every* $a \in c_0^+$ *there is an* $X \in \mathscr{A}$ *with* $\sum_{n\in X} a(n) < +\infty\}$.

$\mathfrak{b} = \min\{|\mathscr{C}|\colon \mathscr{C} \subseteq c_0^+$ *and* $(\forall X \in [\omega]^\omega)(\exists a \in \mathscr{C})(\sum_{n\in X} a(n) = +\infty)\}$.

Proof. Assume $f \in {}^\omega\omega$ and define $a(f) \in c_0^+$ as follows:

$$a(f)(i) = \begin{cases} (\log(k+1))/(k+1) & \text{if } i \in (f(k-1), f(k)\rangle \quad \text{for } k > 0, \\ 1 & \text{if } i \in \langle 0, f(0)\rangle. \end{cases}$$

Assume $X \in [\omega]^\omega$ is such that $\sum_{i\in X} a(f)(i) < +\infty$. Then $e(X) \mathrel{{}^*{\geqslant}} f$ (where $e(X)(n) = \min\{i\colon |\langle 0, i\rangle \cap X| = n\}$). To prove this assume on the contrary that the set $A = \{n\colon e(X)(n) < f(n)\}$ is infinite and that $k \in A$. Then

$$\sum_{0\leqslant i\leqslant k} a(f)(e(X)(i)) \geqslant (k+1)\log(k+1)/(k+1) = \log(k+1) \to +\infty.$$

From this we obtain the $\cdots \leqslant \mathfrak{b}$ and $\cdots \geqslant \mathfrak{d}$ part of the theorem. The proof of this part of the theorem owes much to a result of Copláková-Hartová, which was a part of a preliminary version of [3] but did not appear in the final one.

For the remaining inequalities observe that if for $a \in c_0^+$ we define

$$f(a)(k) = \min\left\{i\colon (\forall j \geqslant i)\left(a(j) < \frac{1}{2^k}\right)\right\} \quad \text{and} \quad f(a) <^* g$$

then $\sum_{i\in\omega} a(g(i)) < +\infty$. □

Problem. Is it consistent with ZFC that $\mathfrak{d} < \mathfrak{u}_r$? (Where

$$\mathfrak{u}_r = \begin{cases} \min\{|\mathscr{B}|\colon \mathscr{B} \text{ is a base of a rapid filter on } \omega\} \\ \qquad \text{if there are rapid filters,} \\ 2^\omega \quad \text{otherwise.} \end{cases}$$

Miller [10] showed that in Laver's model for Borels conjecture [6] there are no rapid filters, but in this model $\mathfrak{d} = 2^\omega = \omega_2$, so this is not the model we are looking for.

Assume (X, τ) is a topological space and define

$$\mathfrak{r}(X) = \begin{cases} \min\{|\mathcal{A}|: \mathcal{A} \subseteq \tau \;\&\; (\forall f \in C(X))(\exists A \in \mathcal{A})(f\restriction A \text{ is constant})\} \\ \qquad \text{if there are such } \mathcal{A}, \\ +\infty \quad \text{otherwise,} \end{cases}$$

$$\mathfrak{s}(X) = \min\{|\mathcal{C}|: \mathcal{C} \subseteq C(X) \;\&\; (\forall A \in \tau)(\exists f \in \mathcal{C})(f\restriction A \text{ is not constant})\}.$$

Clearly $\mathfrak{r}_\sigma = \mathfrak{r}(\omega^*)$ and $\mathfrak{s}_\sigma = \mathfrak{s}(\omega^*)$. We know that for all spaces with the G_δ-property $\mathfrak{r}(X)$ is defined and $\mathfrak{r}(X) \leqslant \pi w(X)$. Are there any other estimations?

Recall [1] that $\mathfrak{h} = \min\{|\mathcal{F}|: (\forall F \in \mathcal{F})(F$ is a nowhere dense subset of $\omega^*)$ & $(\bigcup \mathcal{F}$ is dense in $\omega^*)\}$.

Notice that $\mathfrak{s}_\sigma = \mathfrak{s}$ is a sharpening of $\mathfrak{h} \leqslant \mathfrak{s}_\sigma$ which is implicitly proved in Lemma 2 of [8] and that the very construction of [8] works under a slightly weaker assumption, namely the following holds: If $\mathfrak{s} = \mathfrak{b} = 2^\omega$ then there is a $\{0, 1\}$-sequentially regular Fréchet space which is sequentially complete but fails to be $\{0, 1\}$-sequentially complete. This is still only a partial answer to a question from [5] and it is still open and interesting whether such a construction can be exhibited naively, i.e., in ZFC.

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