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MORE ON SET-THEORETIC CHARACTERISTICS OF SUMMABILITY OF SEQUENCES BY REGULAR (TOEPLITZ) MATRICES

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Abstract: We consider set-theoretic characteristics which reflect some properties of summation of sequences by regular matrices (row-submatrices of the diagonal matrix respectively) acting on ${}^{\omega}2$ and 1^{ω}, and we give some relations between them. We improve the lower bound for the minimal size of a family of regular matrices such that every bounded sequence of real numbers is summed by one of them.

Key words: Cardinal characteristics, matrix summation.

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§ 1. Introduction, notation and results

1.1. Introduction. Recently V.I. Malychin and M.N.Čholščevnikova discovered that some problems related to the summation methods (for sequences) are set-theoretically sensitive (see [5]). In [6] we introduced cardinal characteristics involved in these problems and gave some estimates using well-known cardinal characteristics of $\mathcal{P}(\omega)$ and the Baire space ${}^{\omega}\omega$ - the value of which depends on the model (additional axiom) of set theory you consider.

In the present paper we improve one result of [6], namely, we improve the lower bound for the minimal size of a family of regular matrices such that every bounded sequence is summed by one of them. Moreover we introduce a few cardinal characteristics which reflect properties of summation of sequences by an arbitrary class $\mathcal{S}$ of regular matrices acting on a subspace X of 1^{∞}. We discuss the extremal cases when $\mathcal{S}$ is the whole class of regular matrices or $\mathcal{S}$ is the class of row-submatrices of the diagonal regular matrix, and $X=1^{\infty}$ or $X={}^{\omega}2$.

1.2. Notation and what is already known. We use the standard set-theoretic notation (see e.g. [3]).

As a rule, ω denotes the set of all natural numbers, ${}^{x}y$ denotes the set of all mappings from x to y, 1^{∞} is the set of all bounded sequences of real numbers, $[\varkappa]^{\lambda}=\{X\subseteq\varkappa:|X|=\lambda\}$, $\exists^{\infty}_{n}$ means "there are infinitely many n's" and $\forall^{\infty}_{n}$ means "for all but finitely many n's", $x\subseteq^{*}y$ denotes $x-y$ is finite and for $f,g\in{}^{\omega}\omega$, $f<^{*}g$ denotes $(\forall^{\infty}_{n})(f(n)<g(n))$, $rng(f)=\{f(n):n\in\omega\}$, $[f(n),f(n+1))=\{i\in\omega:f(n)\leq i<f(n+1)\}$.

Let $A=\{a(n,k):n\in\omega, k\in\omega\}$ be a matrix of real numbers. For $b\in{}^{\omega}R$ put $(A,b)(n)=\Sigma\{a(n,k),b(k):0\leq k<+\infty\}$. If $\lim_{n\to\infty}(A,b)(n)$ exists, it is called the A-limit of b. Denote $R(A)=\{b\in l^{\infty}:A\text{-}\lim_{n\to\infty} b(n)$ exists$\}$. We say that A is regular (or also Toeplitz, see [1]) if the following three conditions are satisfied:

(a) $\exists m\forall^{\infty}_{n}\ \Sigma\{|a(n,k)|:0\leq k<+\infty\}<m$,

(b) $\forall k\lim_{n\to\infty} a(n.k)=0$,

(c) $\Sigma\{a(n,k):0\leq k<+\infty\}=c(n)\longrightarrow 1$ as $n\longrightarrow+\infty$.

Denote by $\mathcal{M}$ the set of all regular matrices. Recall that if $\lim_{k\to\infty} b(k)=x$ then $A\text{-}\lim_{k\to\infty} b(k)=x$ for all $A\in\mathcal{M}$. Denote $Mon({}^{\omega}\omega)=\{f\in{}^{\omega}\omega:n<m$ implies $f(n)<f(m)\}$; for $f\in Mon({}^{\omega}\omega)$ let $I(f)$ denote the matrix $\{a(n,k):n\in\omega,k\in\omega\}$ such that $a(n,k)=1$ iff $k=f(n)$ and $a(n,k)=0$ iff $k\neq f(n)$. Let $\mathcal{D}=\{I(f):f\in Mon({}^{\omega}\omega)\}$. Notice that $\mathcal{D}\subseteq\mathcal{M}$. For $\mathcal{S}\subseteq\mathcal{M}$ and $X\subseteq l^{\infty}$ put

$$\mathcal{R}(\mathcal{S},X)=\{Y\subseteq X:(\exists A\in\mathcal{S})(Y\subseteq R(A))\},$$
$$Cov(\mathcal{S},X)=\min\{|\mathcal{A}|:\mathcal{A}\subseteq\mathcal{S}\text{ and }\bigcup\mathcal{R}(\mathcal{A},X)=X\},$$

and $Non(\mathcal{S},X)=\min\{|Y|:Y\subseteq X$ and $Y\notin\mathcal{R}(\mathcal{S},X)\}$. Note that J(Cov(J),Non(J) resp.) of [6] is equal to $\mathcal{R}(\mathcal{M},l^{\infty})$ ($Cov(\mathcal{M},l^{\infty})$, $Non(\mathcal{M},l^{\infty})$ resp.).

Let

$\underline{b}=\min\{|\mathcal{B}|:\mathcal{B}\subseteq{}^{\omega}\omega$ and $(\forall f\in{}^{\omega}\omega)(\exists g\in\mathcal{B})(\exists^{\infty}_{n})(g(n)>f(n))\}=$
$=\min\{|\mathcal{B}|:\mathcal{B}$ is an unbounded family in $({}^{\omega}\omega,<^{*})\}$

$\underline{d}=\min\{|\mathcal{D}|:\mathcal{D}\subseteq{}^{\omega}\omega$ and $(\forall f\in{}^{\omega}\omega)(\exists g\in\mathcal{D})(\forall^{\infty}_{n})(g(n)>f(n))\}=$
$=\min\{|\mathcal{D}|:\mathcal{D}$ is a dominating family in $({}^{\omega}\omega,<^{*})\}$

and

$\underline{s}=\min\{|\mathcal{S}|:\mathcal{S}\subseteq[\omega]^{\omega}$ and $(\forall X\in[\omega]^{\omega})(\exists S\in\mathcal{S})(|X\cap S|=|X-S|=\aleph_0)\}=$
$=\min\{|\mathcal{S}|:\mathcal{S}$ is a splitting family on $\omega\}$

(see [vD]). It was proved in [6] that $\underline{b}\leq Cov(\mathcal{M},l^{\infty})$ and $\underline{s}\leq Non(\mathcal{M},l^{\infty})\leq\underline{b}.\underline{s}$ and in [5] the consistency of "ZFC+$Cov(\mathcal{M},l^{\infty})<2^{\omega}$" was prov ed.

1.3. Results. We say that a family $\mathcal{A}\subseteq[\omega]^{\omega}$ is an <u>attractive family for</u> $X\subseteq l^{\infty}$ if for every $c\in X$ there is an $R\in\mathcal{A}$ such that $\lim\{c(n):n\in R\}$ does exist. We say that a family $\mathcal{C}\subseteq X\subseteq l^{\infty}$ is <u>chaotic</u> if for every $R\in[\omega]^{\omega}$ there is a $c\in\mathcal{C}$ such that $\lim\{c(n):n\in R\}$ does not exist (see [7]). Notice that $\underline{s}=\min\{|\mathcal{C}|:\mathcal{C}\subseteq{}^{\omega}2$ is a chaotic family$\}$. Define

$\underline{r}=\min\{|\mathcal{A}|: \mathcal{A}$ is an attractive family for ${}^{\omega}2\}$

$\underline{s}_{\sigma}=\min\{|\mathcal{C}|: \mathcal{C}\subseteq 1^{\infty}$ is a chaotic family$\}$

$\underline{r}_{\sigma}=\min\{|\mathcal{A}|: \mathcal{A}$ is an attractive family for $1^{\infty}\}$.

These numbers were studied in [7] in their own nature as cardinal characteristics of $\omega^* = \beta\omega - \omega$ and $\underline{s}=\underline{s}_{\sigma}$ was proved.

We prove

Theorem 1. $\underline{s}=\mathrm{Non}(\mathcal{D}, {}^{\omega}2)$,

$\underline{s}_{\sigma}=\mathrm{Non}(\mathcal{D}, 1^{\infty})$,

$\underline{r}=\mathrm{Cov}(\mathcal{D}, {}^{\omega}2)$,

$\underline{r}_{\sigma}=\mathrm{Cov}(\mathcal{D}, 1^{\infty})$.

As a corollary of the mentioned result $\underline{s}=\underline{s}_{\sigma}$ from [7] we obtain $\mathrm{Non}(\mathcal{D}, 1^{\infty})= =\mathrm{Non}(\mathcal{D}, {}^{\omega}2)$. The following problem arose naturally:

Problem. Is $\mathrm{Non}(\mathcal{M}, 1^{\infty})=\mathrm{Non}(\mathcal{M}, {}^{\omega}2)$ provable in ZFC ?

By a detailed inspection of proofs of [6] and [5] we easily find out that the following holds: $\mathrm{Mon}(\mathcal{M}, {}^{\omega}2)\leq \underline{b}.\underline{s}$ and $\underline{b}\leq\mathrm{Cov}(\mathcal{M}, {}^{\omega}2)$. We prove the second inequality in

Theorem 2. $\min(\underline{r},\underline{d})\leq\mathrm{Cov}(\mathcal{M}, {}^{\omega}2)$.

The situation between the considered cardinal characteristics can be described now by the following diagrams, where $\longrightarrow$ means that $\leq$ is provable in ZFC.

$$\begin{array}{ccccc} \min(\underline{r},\underline{d}) & \longrightarrow & \mathrm{Cov}(\mathcal{M}, {}^{\omega}2) & \longrightarrow & \underline{r}=\mathrm{Cov}(\mathcal{D}, {}^{\omega}2) \\ & & \searrow & & \searrow \\ & & \mathrm{Cov}(\mathcal{M}, 1^{\infty}) & \longrightarrow & \underline{r}_{\sigma}=\mathrm{Cov}(\mathcal{D}, 1^{\infty}) \end{array}$$

$$\underline{s}=\underline{s}_{\sigma}=\mathrm{Non}(\mathcal{D}, 1^{\infty})=\mathrm{Non}(\mathcal{D}, {}^{\omega}2)\longrightarrow\mathrm{Non}(\mathcal{M}, 1^{\infty})\longrightarrow\mathrm{Non}(\mathcal{M}, {}^{\omega}2)\longrightarrow\underline{b}.\underline{s}$$

Easily $\underline{b}\leq\min(\underline{r},\underline{d})$ and that the improvement of Theorem 2 is substantial is shown by

Theorem 3. Con(ZFC + "$\underline{b}<\min(\underline{r},\underline{d})$").

§2. Proofs of inequalities

2.1. Proof of Theorem 1. Take $f\in\mathrm{Mon}({}^{\omega}\omega)$ and $x\in{}^{\omega}2$. Observe that $(I(f).x)(n)=x(f(n))$, therefore $I(f)\text{-}\lim_{n\to\infty} x(n)$ exists iff $\lim\{x(n):n\in \mathrm{rng}(f)\}$ exists and moreover $\mathrm{Mon}({}^{\omega}\omega)$ are exactly increasing enumerations of infinite subsets of ω. Keeping this in mind we easily get

$\mathrm{Non}(\mathcal{D},X)=\min\{|Y|:Y\subseteq X$ and $Y\notin\mathcal{R}(\mathcal{D},X)\}=$
$=\min\{|Y|:Y\subseteq X$ and $(\forall A\in\mathcal{D})(\exists y\in Y)\ A-\lim_{n\to\infty}y(n)$ does not exist$\}=$
$=\min\{|Y|:Y\subseteq X$ and $(\forall f\in\mathrm{Mon}({}^{\omega}\omega))(\exists y\in Y)\ \lim\{y(n):n\in \mathrm{rng}(f)\}$ does not exist$\}$ $=\min\{|Y|:Y\subseteq X$ and $(\forall Z\in[\omega]^{\omega})(\exists y\in Y)\ \lim\{y(n):n\in Z\}$ does not exist$\}$ $=\min\{|Y|:Y\subseteq X$ and Y is a chaotic family$\}$. Especially, $\mathrm{Non}(\mathcal{D},{}^{\omega}2)=\underline{s}$ and $\mathrm{Non}(\mathcal{D},l^{\infty})=\underline{s}_{\sigma}$. $\mathrm{Cov}(\mathcal{D},X)=\min\{|\mathcal{A}|:\mathcal{A}\subseteq\mathcal{D}$ and $\bigcup\mathcal{R}(\mathcal{A},X)=$ $=X\}$ $=\min\{|\mathcal{A}|:\mathcal{A}\subseteq\mathcal{D}$ and $(\forall c\in X)(\exists A\in\mathcal{A})(A-\lim_{n\to\infty}c(n)$ exists$\}=$
$=\min\{|\mathcal{F}|:\mathcal{F}\subseteq\mathrm{Mon}({}^{\omega}\omega)$ and $(\forall c\in X)(\exists f\in\mathcal{F})(\lim\{c(n):n\subseteq \mathrm{rng}(f)\}$ exists$\}=$
$=\min\{|\mathcal{A}|:\mathcal{A}\subseteq[\omega]^{\omega}$ and $(\forall c\in X)(\exists A\subseteq\mathcal{A})(\lim\{c(n):n\in A\}$ exists$\}=$
$=\min\{|\mathcal{A}|:\mathcal{A}$ is an attractive family for $X\}$. Especially, $\mathrm{Cov}(\mathcal{D},{}^{\omega}2)=\underline{r}$ and $\mathrm{Cov}(\mathcal{D},l^{\infty})=\underline{r}_{\sigma}$.

2.2. Proof of Theorem 2. Assume $\varkappa<\min(\underline{r},\underline{d})$ is a cardinal number and $\mathcal{A}=\{A_{\alpha}:\alpha<\varkappa\}$ is a system of regular matrices. We show that $\bigcup\mathcal{R}(\mathcal{A},{}^{\omega}2)\neq$ $\neq{}^{\omega}2$ i.e. there is a $z\in{}^{\omega}2$ such that for every $\alpha<\varkappa$ the $A_{\alpha}-\lim_{n\to\infty}z(n)$ does not exist.

For every matrix A_{α} there is a row-submatrix B_{α} and a function $l_{\alpha}\in\mathrm{Mon}({}^{\omega}\omega)$ such that for every $z\in{}^{\omega}2$ and $n\in\omega$.

$(*)\quad [l_{\alpha}(n),l_{\alpha}(n+1))\subseteq z^{-1}(0)$ implies $(B_{\alpha}.z)(n)<1/4$

and

$(**)\quad [l_{\alpha}(n),l_{\alpha}(m+1))\subseteq z^{-1}(1)$ implies $(B_{\alpha}.z)(n)>3/8$

As $R(A_{\alpha})\subseteq R(B_{\alpha})$, to prove the theorem it suffices to find $z\in{}^{\omega}2$ such that for every $\alpha<\varkappa$ there are infinitely many n's such that $(*)$ holds and there are infinitely many n's such that $(**)$ holds.

Define $g_{\alpha}(n)=l_{\alpha}(n^2)$ for $\alpha<\varkappa$. The family $\{g_{\alpha}:\alpha<\varkappa\}$ is not a dominating family. Take $f\in\mathrm{Mon}({}^{\omega}\omega)$ such that for every $\alpha<\varkappa$ the set $F_{\alpha}=\{n:f(n)>g_{\alpha}(n)\}$ is infinite. For an $n\in F_{\alpha}$ as $g_{\alpha}(n)=l_{\alpha}(n^2)$ then $\bigcup\{[f(i),f(i+1)):i<n\}$ contains n^2-many elements of $\mathrm{rng}(l_{\alpha})$. Therefore the set

$$M_{\alpha}=\{n:|[f(n),f(n+1))\cap \mathrm{rng}(l_{\alpha})|\geq 2\}$$

is infinite for every $\alpha<\varkappa$. The system $\{M_{\alpha}:\alpha<\varkappa\}$ is not an attractive family for ${}^{\omega}2$. Take an $X\in[\omega]^{\omega}$ which emphasizes this, namely for every $\alpha<\varkappa$, $|M_{\alpha}-X|=|M_{\alpha}\cap X|=\aleph_0$ holds. Define

$z(i)=0$ if $i\in[f(n),f(n+1))$ and $n\in X$

and

$z(i)=1$ if $i\in[f(n),f(n+1))$ and $n\notin X$.

Then by ($*$) and ($**$) and properties of f and [illegible] we have

$$z \in \bigcup \{P(B_\alpha): \alpha < \varkappa\}.$$

§ 3. Proof of the consistency

3.1. Some facts about the Cohen extensions. Assume $\varkappa$ is a cardinal number and $N \supseteq M$ is the model of ZFC obtained from M by adding $\varkappa$-many Cohen reals. Then there are $C \in N$ and $B \in N$ where $C: \varkappa \longrightarrow {}^\omega 2$ and $B: \varkappa \longrightarrow {}^\omega\omega$ ($C(\alpha)$, $B(\alpha)$ are called Cohen reals) such that N is the minimal model containing M and C (B respectively). We denote the fact $N=M[C]=M[B]$. Moreover for every $I \in \mathcal{P}(\varkappa) \cap M$ there is a model $M[C|I]=M[B|I]$, the least one containing the restrictions $C|I: I \longrightarrow {}^\omega 2$ and $B|I: I \longrightarrow {}^\omega\omega$ (especially $M[C|\emptyset]=M$). All models $M[C|I]$ have the same cardinal numbers as M has.

For every $\alpha < \varkappa - I$, $C(\alpha)$($B(\alpha)$ respectively) is a Cohen real over $M[C|I]$ i.e.

(i) $C(\alpha)$ is in every comeager subset of ${}^\omega 2 \cap N$ coded in $M[C|I]$

and

(ii) $B(\alpha)$ is in every comeager subset of ${}^\omega\omega \cap N$ coded in $M[B|I]$

(see Theorem VIII.2.1 of [4]). Observe that necessarily $C(\alpha) \notin M[C|I]$, $B(\alpha) \notin M[B|I]$.

Moreover the Cohen extension possesses the following property (see Lemma VIII.2.2 of [4]):

(iii) If $X \in N$ is such that there is an $S \in M$ with $X \subseteq S$ then there is an $I \in [\varkappa]^{\leq |S|} \cap M$ such that $X \in M[C|I]$.

For our proof we need the following observation: for every $I \in \mathcal{P}(\varkappa) \cap M$, $f \in {}^\omega\omega \cap M[C|I]$ and $R \in [\omega]^\omega \cap M[C|I]$

(iv) the set $\{g \in {}^\omega\omega \cap N: g <^* f\}$ is a meager subset of ${}^\omega\omega \cap N$ coded in $M[C|I]$

and

(v) the set $\{g \in {}^\omega 2: R \subseteq^* g^{-1}(0) \text{ or } R \subseteq^* g^{-1}(1)\}$ is a meager subset of ${}^\omega 2 \cap N$ coded in $M[C|I]$.

3.2. Proof of Theorem 3. Assume M is arbitrary, $\varkappa \geq \omega_2$ and $N=M[C]$ as in Section 3.1. Then in N holds "$\underline{b}=\omega_1 < \omega_2 \leq \min(\underline{r},\underline{d})$".

(a) $N \models \underline{b}=\omega_1$, indeed $B|\omega_1 = \{B(\alpha): \alpha < \omega_1\}$ is unbounded in N. Suppose not, and $f \in N$ is an upper bound for $B|\omega_1$. Then $f \subseteq \omega \times \omega$ and by (iii) there is an $I \in [\varkappa]^{\omega} \cap M$ such that $f \in M[C|I]$. Take $\gamma \in \omega_1 - I$, then $B(\gamma) \notin \{g \in \in N : g <^* f\}$ by (ii) and (iv).

(b) $N \models \underline{d} \geq \omega_2$. Assume not and $\mathcal{D} = \{f_\alpha : \alpha < \omega_1\}$ is a dominating family in N. As $\mathcal{D} \subseteq \omega_1 \times (\omega \times \omega)$ by (iii) there is an $I \in [\varkappa]^{\omega_1} \cap M$ such that $\mathcal{D} \in M[C|I]$. Take a $\beta \in \varkappa - I$. Then there is an $\alpha < \omega_1$ with $B(\beta) <^* f_\alpha$ but this contradicts (ii) and (iv),

(c) $N \models \underline{r} \geq \omega_2$. Similarly, assume not and $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ is an attractive family for ${}^{\omega}2$ in N. Then $\mathcal{A} \subseteq \omega_1 \times \omega$, so by (iii) there is an $I \subseteq [\varkappa]^{\omega_1} \cap M$ such that $\mathcal{A} \in M[C|I]$. Take $\beta \in \varkappa - I$, then there is an $\alpha < \omega_1$ such that either $A_\alpha \subseteq^* (C(\beta))^{-1}(0)$ or $A_\alpha \subseteq^* (C(\beta))^{-1}(1)$ but this contradicts (i) and (v).

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