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SET-THEORETIC CHARACTERISTICS OF SUMMABILITY OF SEQUENCES AND CONVERGENCE OF SERIES

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Abstract. Using cardinal characteristics of the power set of integers $\mathcal{P}(\omega)$ and the Baire space ${}^{\omega}\omega$ we estimate the minimal size of a family of regular (Toeplitz) matrices such that every bounded sequence is summed by one of them and the minimal size of a family of bounded sequences such that there is no regular matrix which sums all of them. We give the exact value of the minimal size of a set of convergent series unbounded in the sense that there is no convergent series converging "slower" (in terms of lower order and remainders) than each series of the very set. It is observed that analogous results can be proved for divergent series and/or dominating families.

Key words: cardinal characteristics, matrix summation, the speed of convergence of series.

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§1. Introduction and results

1.1. Recently V. I. Malychin and N. N. Cholščevnikova discovered that some problems related to the summation methods (for sequences) and the convergence of series of nonnegative real numbers are set-theoretically sensitive (see [10] and [2]).

The aim of the present paper is to introduce cardinal characteristics involved in these problems and to give some estimates using some well-known cardinal characteristics of $\mathcal{P}(\omega)$ and ${}^{\omega}\omega$.

1.2. First, let us introduce the basic notions. We use the standard set-theoretic notation (see e.g. [8]). As a rule, ω denotes the set of all natural numbers, $\forall^\infty n$ means "for all but finitely many n's" and $\exists^\infty n$ means "for infinitely many n's", ${}^x y$ denotes the set of all mappings from x to y (e.g. ${}^\omega\omega$ denotes the Baire space), ℓ^∞ is the set of all bounded sequences of real numbers, $[\omega]^\omega = \{X \subseteq \omega : |X| = \aleph_0\}$. Let $A = \{a(n, k) : n \in \omega, k \in \omega\}$ be a matrix of real numbers. We say that A is <u>regular</u> (Toeplitz) (see [4]) if the following three conditions are satisfied:

(a) $\exists m \; \forall^\infty n \; \sum\{|a(n, k)| : 0 \leq k < +\infty\} < m$,

(b) $\forall k \; \lim_{n \to +\infty} a(n, k) = 0$,

(c) $\sum\{a(n, k) : 0 \leq k < +\infty\} \equiv A_n \to 1$ as $n \to +\infty$.

Denote by $\mathcal{M}$ the set of all regular matrices. For $A \in \mathcal{M}$ and $b \in \ell^\infty$ put $(A.b)(n) = \sum\{a(n, k)b(k) : k \in \omega\}$. Clearly, $A.b \in \ell^\infty$. If $\lim_{n \to \infty} (A.b)(n)$ exists, it is called the A-limit of b . Recall that if $\lim_{k \to \infty} b(k) = x$ then $A - \lim_{k \to +\infty} b(k) = x$ for all $A \in \mathcal{M}$. For $A \in \mathcal{M}$ put

$R(A) = \{b \in \ell^\infty : \lim_{n \to \infty} (A.b)(n) \text{ exists}\}$,

$J = \{X \subseteq \ell^\infty : (\exists A \in \mathcal{M})(X \subseteq R(A))\}$,

$\mathrm{Cov}(J) = \min\{|\mathcal{U}| : \mathcal{U} \subseteq \mathcal{M} \;\&\; \bigcup\{R(A) : A \in \mathcal{U}\} = \ell^\infty\}$, and

$\mathrm{Non}(J) = \min\{|X| : X \subseteq \ell^\infty \;\&\; X \notin J\}$.

Finally, let

$\mathcal{K} = \{b \in \ell^\infty : (\forall n)(b(n) \geq 0) \;\&\; \sum\{b(n) : 0 \leq n < +\infty\} < +\infty\}$

be a set of all convergent series with nonnegative terms. On

$\mathcal{K}$ we can define a partial ordering (see also G. M. Fichtengolz, [7]) as follows:

$$a <_0 b \quad \text{iff} \quad \lim_{n \to \infty} (\sum \{a(k) : n \leq k < +\infty\} / \sum \{b(k) : n \leq k < +\infty\}) = 0 .$$

A subset $U \subseteq P$ of the partially ordered set $(P, <)$ is said to be unbounded if there is no $p \in P$ such that $(\forall u \in U)(u < p)$, a subset $D \subseteq P$ is said to be dominating if for every $p \in P$ there is a $d \in D$ with $p < d$.

Analogously as in [5], we write

$\underline{b}(P,<) = \min \{|U| : U$ is unbounded in $(P,<)\}$, and

$\underline{d}(P,<) = \min \{|D| : D$ is dominating in $(P,<)\}$.

We say that a set $X \in [\omega]^\omega$ is sparse according to an $\mathcal{F} \subseteq {}^\omega\omega$ if $(\forall f \in \mathcal{F})(\forall^\infty n)(|[f(n), f(n+1)) \cap X| \leq 1)$, where $[f(n), f(n+1))$ denotes the (half open) interval of natural numbers ([10], see also [6]; [3]).

The following hypothesis was introduced by P. Erdös and G. Piranian (cf. [6] p. 146, [10]):

(EPH) 'For every $\mathcal{F} \subseteq {}^\omega\omega$ with $|\mathcal{F}| < 2^\omega$, there is an $X \in [\omega]^\omega$ which is sparse according to $\mathcal{F}$.

R. G. Cooke in [4] proved that Cov(J) is infinite and Non(J) $\geq \aleph_1$. V. I. Malychin and N. N. Cholščevnikova proved in [10] and [2] that (EPH) implies Cov(J) $= \underline{b}(\mathcal{K}, <_0) = 2^\omega$; Booth lemma implies (EPH) and Non(J) $= 2^\omega$; all inequalities Non(J) $< 2^\omega$, Cov(J) $< 2^\omega$, $\underline{d}(\mathcal{K}, <_0) < 2^\omega$ are consistent, too.

1.3. For $f, g \in {}^{\omega}\omega$ we define

$$f <^{*} g \quad \text{if} \quad (\forall^{\infty} n)(f(n) < g(n)) .$$

The set ${}^{\omega}\omega$ is partially ordered by $<^{*}$. Denote (see [5])

$$\underline{b} = \underline{b}({}^{\omega}\omega, <^{*}) \quad \text{and}$$

$$\underline{d} = \underline{d}({}^{\omega}\omega, <^{*}) .$$

An $\mathcal{S} \subseteq [\omega]^{\omega}$ is a splitting family if $(\forall A \in [\omega]^{\omega})$ $(\exists S \in \mathcal{S})(|A \cap S| = |A - S| = \aleph_0)$. Recall that

$$\underline{s} = \min \{|\mathcal{S}| : \mathcal{S} \text{ is a splitting family}\} .$$

Let us define

$$\underline{e} = \min \{|\mathcal{F}| : \mathcal{F} \subseteq {}^{\omega}\omega \text{ and there is no } X \in [\omega]^{\omega} \text{ sparse according to } \mathcal{F}\} .$$

Note that $\underline{e}$ is the minimal cardinal number for which EPH fails to be true.

1.4. In this section we list the main results and indicate their proofs (they follow as corollaries of lemmas proved in §§2 and 3).

Theorem 1. (a) $\underline{b} = \underline{e}$

(b) $\underline{b} = \underline{b}(\mathcal{K}, <_0)$.

Proof. (a) follows from Lemma 2.1 and Lemma 2.2.

(b) follows from Lemma 2.3 and the fact that (EPH) implies $\underline{b}(\mathcal{K}, <_0) = 2^{\omega}$.

Corollary. $\underline{b} \leq \mathrm{Cov}(J)$.

Proof. The assertion follows from Theorem 1(a) and "(EPH) implies $\mathrm{Cov}(J) = 2^{\omega}$".

Theorem 2. (a) $\underline{s} \leq \mathrm{Non}(J)$

(b) $\mathrm{Non}(J) \leq \underline{b}.\underline{s}$.

Proof. See Lemma 3.1 and Lemma 3.2.

Theorem 3. Assume there is a κ-Luzin set. Then $\mathrm{Non}(J) \leq \kappa$.

Proof. See Lemma 3.4.

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§2. The speed of the convergence of series

2.1. Lemma. Assume that $\mathcal{F} \subseteq {}^{\omega}\omega$ has no $<^*$-upper bound. Then for every $X \in [\omega]^{\omega}$ there is an $f \in \mathcal{F}$ such that $\sup|X \cap [f(n), f(n+1))| = \aleph_0$.

Proof. Let $X = \{x_0 < x_1 < \ldots < x_n < \ldots\}$. Define $g \in {}^{\omega}\omega$ by putting $g(n) = x_{n^2}$. Since the family $\mathcal{F}$ is unbounded, for some $f \in \mathcal{F}$ the set $\{n \in \omega : g(n) < f(n)\}$ is infinite. If $g(n) < f(n)$, then $|X \cap \bigcup_{i=0}^{n-1} [f(i), f(i+1))| > n^2$ and therefore one of the intervals $[f(i), f(i+1))$ contains at least n points of X.

Corollary. $\underline{e} \leq \underline{b}$.

2.2. Lemma. $\underline{b} \leq \underline{e}$.

Proof. Consider $\mathcal{F} \subseteq {}^{\omega}\omega$ such that $|\mathcal{F}| < \underline{b}$. We show that there is an infinite set sparse according to $\mathcal{F}$. The following useful trick was used (implicitly) by many authors and appears as Lemma 3.16 in [1]. Take an $f \mathrel{{}^*{>}} \mathcal{F}$, and w.l.o.g. we can assume that all $g \in \mathcal{F}$ and f are increasing and greater than $\Delta + 1$ (the diagonal $+ 1$). Define $\bar{f}$ inductively by:

$\bar{f}(0) = 0$, $\bar{f}(n+1) = f(\bar{f}(n))$.

Then for every $g \in \mathcal{F}$ and n_0 sufficiently large to fulfil $(\forall n \geq n_0)(f(n) > g(n))$ we have,

$$g(i) \leq \bar{f}(n) \leq g(i+1) \text{ implies } \bar{f}(n+1) \geq g(i+1) .$$

To prove this, assume on contrary that $g(i+1) > \bar{f}(n+1)$.
But then $\bar{f}(n+1) = f(\bar{f}(n)) > g(\bar{f}(n)) > g(g(i)) \geq g(i+1)$, a contradiction. Consequently the range of $\bar{f}$ is sparse according to $\mathcal{F}$.

The statement of Lemma 2.2 appears also independently in [1] as Proposition 3.17.

2.3. Corollary. $\underline{b} \leq \underline{b}(\mathcal{K}, <_0)$.

Proof. The assertion follows immediately from "EPH $\rightarrow \underline{b}(\mathcal{K}, <_0) = 2^{\omega}$" proved in [2].

In order to prove Theorem 1(b) it suffices to prove

Lemma. $\underline{b}(\mathcal{K}, <_0) \leq \underline{b}$.

Proof. Consider $\mathcal{F} \subseteq {}^{\omega}\omega$ such that $|\mathcal{F}| < \underline{b}(\mathcal{K}, <_0)$ and all elements of $\mathcal{F}$ are increasing. For every $g \in \mathcal{F}$ we define

$$c_g(i) = 0 \quad \text{for} \quad i \in [0, g(0))$$

and

$$c_g(i) = \left(\frac{1}{k(k+1)}\right)\left(\frac{1}{g(k+1) - g(k)}\right) \quad \text{for}$$
$$i \in [g(k), g(k+1)) ,$$

i.e., $\sum \{c_g(i) : g(k) \leq i < +\infty\} = \frac{1}{k}$ for every $k \in \omega$

Take $c \in \mathcal{K}$, an $<_0$-upper bound for all c_g's .

We define

$f(k) = \min \{j : \sum \{c(i) : j \leq i < +\infty\} < \frac{1}{k}\}$. We claim that $f \,{}^{*}\!> \mathcal{F}$. By contradiction, let $g \in \mathcal{F}$ be such that $(\overset{\infty}{\exists} k)(f(k) \leq g(k))$. Then

$(\sum \{c_g(i) : g(k) \leq i < +\infty\} / \sum \{c(i) : g(k) \leq i < +\infty\}) \geq 1$ for infinitely many k .

2.4. Denote

$\mathcal{D} = \{b \in \ell^\infty : (\forall n)(b(n) \geq 0) \,\&\, \sum\{b(n) : 0 \leq n < +\infty\} = +\infty\}$.

On $\mathcal{D}$ we can define a partial ordering $<_0$ as follows:

$$a <_0 b \text{ iff } \lim_{n\to\infty} \left(\sum\{a(k) : 0 \leq k \leq n\} / \sum\{b(k) : 0 \leq k \leq n\}\right) = 0 .$$

Put

$$\underline{b}(\mathcal{D}\downarrow, <_0) = \min\{|B| : B \subseteq \mathcal{D} \,\&\, \neg(\exists a \in \mathcal{D})(\forall b \in B)(a <_0 b)\}$$

and

$$\underline{d}(\mathcal{D}\downarrow, <_0) = \min\{|D| : D \subseteq \mathcal{D} \,\&\, (\forall a \in \mathcal{D})(\exists d \in D)(d <_0 a)\} .$$

If we replace the convergent series by the divergent ones and /or the unbounded families by the dominating ones, then we can analogously prove the corresponding results:

$$\underline{d}(\mathcal{K}, <_0) = \underline{d} , \quad \underline{b}(\mathcal{D}\downarrow, <_0) = \underline{b} , \quad \underline{d}(\mathcal{D}\downarrow, <_0) = \underline{d} .$$

§3. Summability of sequences - generalized limits

3.1. Lemma. $\underline{s} \leq \mathrm{Non}(J)$.

Proof. Consider $\mathcal{B} \subseteq \ell^\infty$ such that $|\mathcal{B}| < \underline{s}$. Let $A \in \mathcal{M}$ be an arbitrary regular matrix. Then the sequence $A.b$ is bounded for every $b \in \mathcal{B}$. For every rational number $q \in Q$ put

$$L_{b,q} = \{n : (A.b)(n) \leq q\} .$$

The family $\{L_{b,q} : b \in \mathcal{B} \,\&\, q \in Q\}$ has size $|\mathcal{B}| . \aleph_0 < \underline{s}$, so it cannot be splitting and hence there is an $X \in [\omega]^\omega$ such that for each $b \in \mathcal{B}$ and $q \in Q$ either $X \cap L_{b,q}$ or $X - L_{b,q}$ is finite. The row submatrix

$$A_X = \{a(n, k) : n \in X \,\&\, k \in \omega\}$$

sums all b's . Suppose not, i.e. there is a $b \in \mathcal{B}$ such that $A_X.b$ has two accumulation points, say $u < v$. Take

$q \in Q$ such that $u < q < v$. Then

$$|X \cap L_{b,q}| = |X \cap (\omega - L_{b,q})| = \aleph_0 ,$$

a contradiction.

3.2. Lemma. Non(J) $\leq$ b.s .

Proof. We give a construction of b.s many elements of ${}^{\omega}2$ which cannot be covered by a regular matrix. Namely, let $\mathcal{F} \subseteq {}^{\omega}\omega$ be an unbounded family consisting of increasing functions of size $|\mathcal{F}| = \underline{b}$ and let $\mathcal{S} \subseteq [\omega]^{\omega}$ be a splitting family of size $|\mathcal{S}| = \underline{s}$. For $f \in \mathcal{F}$ and $S \in \mathcal{S}$ let $g_{f,s} \in {}^{\omega}\omega$ be defined by

$$g_{f,s}(k) = 0 \quad \text{if } k \in [f(n), f(n+1)) \text{ and } n \in S$$
$$= 1 \quad \text{if } k \in [f(n), f(n+1)) \text{ and } n \notin S .$$

We need to prove that the set $\{g_{f,s} : f \in \mathcal{F} \text{ and } S \in \mathcal{S}\}$ cannot be covered by a regular matrix.

Fix a regular matrix $A = \{a(n, k) : n \in \omega , k \in \omega\}$. For each n define two numbers l_n, r_n as follows:

$$l_n = \sup \{k : \sum\{a(n, i) : 0 \leq i \leq k\} < \tfrac{1}{8}\} \text{ if it is finite,}$$
$$= 0 \text{ otherwise;}$$
$$r_n = \min \{k : \sum\{a(n, i) : k < i < +\infty\} < \tfrac{1}{8}\} .$$

Obviously $l_n < r_n$, and we may assume that for each n we have $r_n < l_{n+1}$ (take a submatrix if necessary). By Lemma 2.1 there exists $f \in \mathcal{F}$ such that the set

$$M = \{n : [f(n), f(n+1)) \text{ contains at least } 2\ l_j\text{'s}\}$$

is infinite. By the splitting property there exists $S \in \mathcal{S}$ such that $|M \cap S| = |M - S| = \aleph_0$. Consider the function $g_{f,s}$. There exists $j_0 \in \omega$ such that for all $j > j_0$, $9/8 >$

$> \sum_{i=0}^{\infty} a(j, i) > 7/8$.

But now suppose that for some n, $j > j_0$,

$$l_j, l_{j+1} \in [f(n), f(n+1)) .$$

If $n \in S$, then for all $i \in [f(n), f(n+1))$, $g_{f,s}(i) = 0$, in particular, $g_{f,s}(i) = 0$ for all $i \in [l_j, r_j]$. Therefore, for this j we have

$$\sum a(j, i) g_{f,s}(i) \leq \sum_{i=0}^{l_j} a(j, i) + \sum_{i=r_j+1}^{+\infty} a(j, i) \leq \frac{1}{4} .$$

On the other hand, if $n \notin S$, then

$$\sum a(j, i) g_{f,s}(i) > \frac{7}{8} - 2.\frac{1}{8} = \frac{5}{8} .$$

But this implies $\limsup_{j \to \infty} \sum a(j, i) g_{f,s}(i) \geq \frac{5}{8} > \frac{1}{4} \geq$

$\geq \liminf_{j \to \infty} \sum a(j, i) g_{f,s}(i)$.

3.3. To prove the Theorem 3 we need the following

Lemma. For every regular matrix A the set

$$R(A) \cap {}^{\omega}2 \text{ is meager.}$$

Proof. Take any $A \in \mathcal{M}$. Consider ${}^{<\omega}2 = \bigcup \{{}^n2 : n \in \omega\}$ with the partial ordering $\subseteq$. Put

$$\bar{D}_n = \{p \in {}^{<\omega}2 : (\exists i > n)(\sum\{a(i, k).p(k) : k \in \mathrm{dom}(p)\} > \frac{3}{4} \ \& \ \& \sum\{|a(i, k)| : \mathrm{dom}(p) \leq k < +\infty\} < \frac{1}{12})\}$$

and

$$\underline{D}_n = \{p \in {}^{<\omega}2 : (\exists i > n)(\sum\{a(i, k).p(k) : k \in \mathrm{dom}(p)\} < \frac{1}{4} \ \& \ \& \sum\{|a(i, k)| : \mathrm{dom}(p) \leq k < +\infty\} < \frac{1}{12})\},$$

where $\mathrm{dom}(p)$ denotes the domain of p, a natural number.

Using properties of regular matrices we easily see that both families are dense in $({}^{<\omega}2, \subseteq)$. For $s \in {}^{<\omega}2$, put $[s] = \{f \in {}^{\omega}2 : s \subseteq f\}$. $\overline{C}_n = \bigcup\{[s] : s \in \overline{D}_n\}$ and $\underline{C}_n = \bigcup\{[s] : s \in \underline{D}_n\}$ are open dense subsets of $\langle 0, 1\rangle$ and so is also $C_n = \overline{C}_n \cap \underline{C}_n$. Observe that if $x \in C_n$, then

$(\exists i > n)(\sum\{a(i, k).x(k) : 0 \leq k < +\infty\} > \frac{2}{3})$ and

$(\exists j > n)(\sum\{a(j, k).x(k) : 0 \leq k < +\infty\} < \frac{1}{3})$.

But then $R(A) \subseteq \bigcup_{n\in\omega}(\langle 0, 1\rangle - C_n)$, which proves the lemma.

3.4. A set $L \subseteq \langle 0, 1\rangle$ is said to be a κ-Luzin set (see e.g. [9]) if $|L| \geq \kappa$ and for every meager set $X \subseteq \langle 0, 1\rangle$ we have $|X \cap L| < \kappa$.

Lemma. Assume there is a κ-Luzin set. Then $\mathrm{Non}(J) \leq \kappa$.

Proof. Let L be a κ-Luzin set. W.l.o.g. we can assume that $|L| = \kappa$ (else an arbitrary κ-sized subset of L is also a κ-Luzin set). As $R(A) \cap {}^{\omega}2$ is meager for every regular matrix A, necessarily $L \setminus R(A) \neq \emptyset$.

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