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The strength of the comparison test versus gaps between convergent and divergent series

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Abstract. We consider series of nonnegative real numbers ordered under eventual dominance (comparison). We prove that the minimal size of a comparison family of convergent series equals to the minimal size of a base of the ideal of Lebesgue measure zero sets. We mention a connection with another comparison test which uses the ordering according to the speed of convergence of remainders. We prove that there is an (ω_1, ω_1^*)-gap between convergent and divergent series.

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1. Introduction. In this paper we consider set-theoretic characteristics connected with the convergence or divergence of series consisting of nonnegative real numbers. In K. T. Smith's textbook ([8], p. 94) we can read: "The comparison test is probably the most valuable convergence test there is - but plainly the value depends on having a large stock of series to use in comparison". We show that to decide the convergence of all convergent series we need this stock as large as the minimal base of the ideal of Lebesgue measure zero sets. This number, depending on the additional axioms of set theory, can be different but in general it ranks among the greatest cardinal characteristics of the real line. We mention the connection with another comparison test which uses the ordering according to the speed of convergence of remainders we considered in [9]. On the other side, we found (personally) interesting that independently of the axioms of set theory there are (ω_1, ω_1^*)-gaps between convergent and divergent series.

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2. The strength of comparison tests. Our notation is standard set-theoretic one, see e.g. [6]. Let $(P, <)$ be a partial ordering; $B \subseteq P$ is said to be unbounded if $(\forall p \in P)(\exists b \in B)(b \nleq p)$, $D \subseteq P$ is said to be dominating if $(\forall p \in P)(\exists d \in D)(p < d)$. We define (the notation is motivated by the unified one in [2]):

$\underline{b}(P, <) = \min \{|B| : B$ is unbounded in $(P, <)\}$,

$\underline{d}(P, <) = \min \{|D| : D$ is dominating in $(P, <)\}$.

Further ${}^x y$ is the set of all functions from x to y, ω is the set of natural numbers, $c_0^+ = \{a \in {}^\omega\langle 0, +\infty) : \lim \{a(n) : n \in \omega\} = 0$, $\mathcal{K} = \{a \in c_0^+ : \sum \{a(n) : n \in \omega\} < +\infty\}$ and $\mathcal{Z} = \{a \in c_0^+ : \sum \{a(n) : n \in \omega\} = +\infty\}$. We define a partial ordering as follows: for $a, b \in c_0^+$ put $a <^* b$ if $\{n : a(n) \geq b(n))\}$ is finite; $x \subseteq^* y$ denotes that $x - y$ is finite. For $\mathbb{L} = \{X \subseteq \mathrm{Real}$: the Lebesgue measure $\mu(X) = 0\}$, the additivity of measure $\mathrm{Add}(\mathbb{L})$ is defined as $\min \{|\mathcal{U}| : \mathcal{U} \subseteq \mathbb{L}\ \&\ \bigcup \mathcal{U} \notin \mathbb{L}\}$. We say that $\mathcal{E} \subseteq \mathbb{L}$ is a base of $\mathbb{L}$ if $(\forall X \in \mathbb{L})(\exists E \in \mathcal{E})(X \subseteq E)$ and $\Delta^*(\mathbb{L}) = \min \{|\mathcal{E}| : \mathcal{E} \subseteq \mathbb{L}$ is a base of $\mathbb{L}\}$. T. Bartoszynski proved in [1] that $\mathrm{Add}(\mathbb{L}) = 2^\omega$ if and only if $\underline{b}(\mathcal{K}, <^*) = 2^\omega$. We prove the following theorem:

Theorem 1. $\underline{d}(\mathcal{K}, <^*) = \Delta^*(\mathbb{L})$.

Proof. Case $\underline{d}(\mathcal{K}, <^*) \leq \Delta^*(\mathbb{L})$. The following lemma appears as Lemma 7 in [3], but the idea appeared essentially already in [1].

Lemma 1. (T. Bartoszynski, D. H. Fremlin). Let $\mathcal{S} = \{S \subseteq \omega \times \omega : (\forall i \in \omega)(|\{j : (i, j) \in S| \leq (i + 1)^2)\}$. Then there are functions $f \mapsto V_f : {}^\omega\omega \to \mathbb{L}$ and $E \mapsto R_E : \mathbb{L} \to \mathcal{S}$ such that $V_f \subseteq E$ implies $f \subseteq^* R_E$.

Corollary. Assume $\mathcal{E} \subseteq \mathbb{L}$ is a base of $\mathbb{L}$. Then there is a system $\{R_E : E \in \mathcal{E}\} \subseteq \mathcal{S}$ such that $(\forall f \in {}^\omega\omega)(\exists E \in \mathcal{E})(f \subseteq^* R_E)$.

We note that putting $f_E(n) = \max \{i + 1 : (n, i) \in R_E\}$ we obtain a dominating family in $({}^\omega\omega, <^*)$ of size $|\mathcal{E}|$ which proves $\underline{d} \leq \Delta^*(\mathbb{L})$, originaly proved in [7] ($\underline{d} = \underline{d}({}^\omega\omega, <^*)$ - see [2]).

To continue the proof of our inequality, we need the following notation: Q is the set of rational numbers, for $g \in {}^\omega\omega$ put $\langle g(n), g(n+1)) = \{i \in \omega : g(n) \leq i < g(n+1)\}$,
$\langle g(n), g(n+1))_Q = \{f : f$ is a mapping of $\langle g(n), g(n+1))$ into $Q\}$,

$\mathcal{X}_g = \prod_{n\in\omega} \langle g(n), g(n+1))_Q$,

$\mathcal{Y}_g = \{R \subseteq \bigcup_{n\in\omega} \{n\} \times \langle g(n), g(n+1))_Q : (\forall i \in \omega)(|\{h \in \langle g(n), g(n+1))_Q : (i, h) \in R\}| \leq (i+1)^2)\}$,

if moreover $a \in \mathcal{K} \cap {}^{\omega}Q$, then

$f_a(n) = \min \{i \in \omega : \sum \{a(j) : i \leq j < +\infty\} < 1/2^n\}$,

$a_g(n) = a \upharpoonright \langle g(n), g(n+1))$, i.e. $a_g \in \mathcal{X}_g$.

For $R \in \mathcal{Y}_g$ put $a_R(i) = \max \{h(i) : (n, h) \in R \;\&\; \sum \{h(j) : g(n) \leq j < g(n+1)\} < 1/2^n\}$; clearly $a_R \in \mathcal{K} \cap {}^{\omega}Q$.

The technical part of the proof of (ii) $\Rightarrow$ (i) in Lemma of [1] can be reformulated as follows:

Lemma 2. Assume $f \in {}^{\omega}\omega$, $a \in \mathcal{K} \cap {}^{\omega}Q$ and $R \in \mathcal{Y}_g$ are such that $f_a <^* g$ and $a_g \subseteq^* R$. Then $a <^* a_R$.

The proof is straightforward. To prove the inequality we apply Corollary to the space $\mathcal{X}_g$. Assume $\mathcal{C} \subseteq {}^{\omega}\omega$ is a dominating family in $({}^{\omega}\omega, <^*)$ and $\mathcal{E} \subseteq \mathbb{L}$ is a base of $\mathbb{L}$ such that $|\mathcal{C}| = \underline{d}$ and $|\mathcal{E}| = \Delta^*(\mathbb{L})$. Then for every $g \in \mathcal{C}$ there is a system $\{R_E^g : E \in \mathcal{E}\} \subseteq \mathcal{Y}_g$ such that $(\forall f \in \mathcal{X}_g)(\exists E \in \mathcal{E})$ $(f \subseteq^* R_E^g)$. Now, applying Lemma 2, we see that the system $\{a(R_E^g) : g \in \mathcal{C} \;\&\; E \in \mathcal{E}\}$ is a dominating family in $(\mathcal{K}, <^*)$ of size $\underline{d}$. $\Delta^*(\mathbb{L}) = \Delta^*(\mathbb{L})$. For, take any $b \in \mathcal{K}$. Then there is an $a \in \mathcal{K} \cap {}^{\omega}Q$ such that $b <^* a$. Take $g \in \mathcal{C}$ such that $f_a <^* g$. Then $a_g \in \mathcal{X}_g$ and let $E \in \mathcal{E}$ be such that $a_g \subseteq^* R_E^g$. Finally $b <^* a <^* a(R_E^g)$.

<u>Case $\Delta^*(\mathbb{L}) \leq \underline{d}(\mathcal{K}, <^*)$</u> . Fix a dominating family $\mathcal{A} \subseteq \mathcal{K}$ in $(\mathcal{K}, <^*)$ and an enumeration $\{I_n : n \in \omega\}$ of all intervals of real numbers with rational endpoints. Put $E_a = \bigcap_n \bigcup \{I_m : m \geq n \;\&\; a(m) > \mu(I_m)\}$. We show that $\{E_a : a \in \mathcal{A}\}$ is a base of $\mathbb{L}$. Take $X \in \mathbb{L}$, then there is a $y \in {}^{\omega}2$ such that $X \subseteq \mathcal{U}_y = \bigcap_n \bigcup \{I_m : m \geq n \;\&\; y(m) = 1\}$ and $\sum \{y(n) . \mu(I_n) : n \in \omega\} < +\infty$. Take $a \in \mathcal{A}$ such that $a(n) > y(n) . \mu(I_n)$ holds for all but finitely many n . Then $X \subseteq \mathcal{U}_y \subseteq E_a$.

There is an interesting feature of our theorem. We can compare the strength of two different comparison tests. For $a, b \in \mathcal{K}$ we define $a <_0 b$ if $\lim_{n\to\infty} (\sum \{a(i) : n \leq i < +\infty\} / \sum \{b(i) : n \leq i < +\infty\}) = 0$. We proved in [9] that

$\underline{d}(\mathcal{K}, <_0) = \underline{d}$. As $\underline{d} \leq \Delta^*(\mathbb{L})$ we immediately obtain

Observation. $\underline{d}(\mathcal{K}, <_0) \leq \underline{d}(\mathcal{K}, <^*)$;

freely rephrased: the comparison test using the ordering $<_0$ is stronger than the one using $<^*$.

Notice that $d(\mathcal{D}\downarrow, <^*) = 2^\omega$ and $b(\mathcal{D}\downarrow, <^*) = 2$, where

$\underline{b}(\mathcal{D}\downarrow, <^*) = \min\{|B| : B \subseteq \mathcal{D} \ \& \ \neg(\exists a \in \mathcal{D})(\forall b \in B)(a <^* b)\}$

and $\underline{d}(\mathcal{D}\downarrow, <^*) = \min\{|D| : D \subseteq \mathcal{D} \ \& \ (\forall a \in \mathcal{D})(\exists d \in D)(d <^* a)\}$.

The corresponding results for $<_0$ are $\underline{b}(\mathcal{D}\downarrow, <_0) = \underline{b}$ and $\underline{d}(\mathcal{D}\downarrow, <_0) = \underline{d}$ (see [9]).

3. Gaps between $\mathcal{K}$ and $\mathcal{D}$. In addition to the terminology introduced in §2, we need the following one. Let $(P, <)$ be a partial ordering, $A = \{a_\alpha : \alpha < \omega_1\} \subseteq P$ and $B = \{b_\alpha : \alpha < \omega_1\} \subseteq P$ be such that $(\forall \alpha < \beta < \omega_1)(a_\alpha < a_\beta < b_\beta < b_\alpha)$. The pair (A, B) is said to be an (ω_1, ω_1^*)-gap if there is no $c \in P$ with $a_\alpha < c < b_\alpha$ for every $\alpha < \omega_1$. The gap (A, B) in $(c_0^+, <^*)$ is a gap between $\mathcal{K}$ and $\mathcal{D}$ if $A \subseteq \mathcal{K}$ and $B \subseteq \mathcal{D}$.

Theorem 2. There is an (ω_1, ω_1^*)-gap between $\mathcal{K}$ and $\mathcal{D}$ under $<^*$.

Proof. We follow the original proof of F. Hausdorf ([4]). For $a, b \in c_0^+$ put $K(a, b) = \min\{n : (\forall k \geq n)(a(k) < b(k))\}$. For $A \subseteq c_0^+$ and $b \in c_0^+$ such that $(\forall a \in A)(a <^* b)$ we say that b is close to A (denote $A \curlyvee b$) if $(\forall k \in \omega)(\{a \in A : K(a, b) \leq k\}$ is finite). By transfinite induction we construct $A = \{a_\alpha : \alpha < \omega_1\} \subseteq \mathcal{K}$ and $B = \{b_\alpha : \alpha < \omega_1\} \subseteq \mathcal{D}$ such that for every $\alpha < \beta < \omega_1$

(1) $a_\alpha <^* a_\beta <^* b_\beta <^* b_\alpha$

and (2) $\{a_\xi : \xi < \alpha\} \curlyvee b_\alpha$ holds. Then (A, B) is a gap between $\mathcal{K}$ and $\mathcal{D}$. To proceed by induction, we have to verify two properties:

1. Let $A = \{a_n : n \in \omega\} \subseteq \mathcal{K}$ and $B = \{b_n : n \in \omega\} \subseteq \mathcal{D}$ be such that $(\forall m < n < \omega)(a_m <^* a_n <^* b_n <^* b_m)$. Then there is an $a \in \mathcal{K}$ and $b \in \mathcal{D}$ such that for every $n \in \omega$ we have $a_n <^* a <^* b <^* b_n$. For this put $n_0 = 0$ and

$$\begin{aligned} n_k = \min\{n > n_{k-1} : (\forall m \geq n)(&a_k(m) > a_{k-1}(m) \\ &\& \ a_k(m) < b_k(m) \\ &\& \ b_k(m) < b_{k-1}(m) \\ &\& \ \textstyle\sum\{a_k(i) : m \leq i < +\infty\} < 1/2^k \\ &\& \ \textstyle\sum\{b_{k-1}(i) : n_{k-1} \leq i < m\} > 1/k)\} . \end{aligned}$$

Define $a(i) = a_k(i)$ and $b(i) = b_k(i)$ for $i \in \langle n_k, n_{k+1})$.

2. Let $A = \{a_n : n \in \omega\} \subseteq \mathcal{K}$ and $b \in \mathcal{L}$ be such that $(\forall m < n < \omega)(a_m <^* a_n <^* b)$. Then there is an $x \in \mathcal{D}$ such that $(\forall m \in \omega)(a_m <^* x <^* b)$ and $(\forall k \in \omega)(\{n : K(a_n, x) \leq k\}$ is finite). Similarly put $n_0 = 0$ and $n_k = \min\{n : (\forall m \geq n)$ $(a_k(m) < a_{k+1}(m) < b(m)\ \&\ \sum\{b(i) - a_k(i) : n_{k-1} < i < n\} > 2/k\}$. Define $x(i) = (a_k(i) + b(i))/2$ for $i \in (n_{k-1}, n_k)$ and $x_{n_k} =$ $= (a_k(n_k) + a_{k+1}(n_k))/2$.

Analogously, by induction we can prove the following

Observation. Every linear ordering of size ω_1 is embeddable (order preserving) into c_0^+ .

Corollary. If $2^\omega < 2^{\omega_1}$, then there are 2^{ω_1}-many different (ω_1, ω_1^*)-gaps in c_0^+ (where different gaps means that the ideal points in the completion of the partially ordered set $(c_0^+, <^*)$ corresponding to these gaps are different).

Proof. We modify a technique from [5]. Take $D \subseteq 2^{\omega_1}$ a dense subset of size ω_1 . Every $x \in 2^{\omega_1} - D$ corresponds to a Dedekind cut in D . Embedd D into c_0^+ . Only 2^ω-many images of these cuts are filled by some element of c_0^+ . The rest are gaps in c_0^+ , 2^{ω_1}-many of them are (ω_1, ω_1^*)-gaps.

Observe that $\mathcal{K}$ is an ideal in $(c_0^+, <^*)$ and $\mathcal{L}$ is a filter. Moreover $\mathcal{K}$ is such an ideal that the complement $c_0^+ - \mathcal{K} = \mathcal{D}$ is a filter. This led us to a natural question:

Consider $B = \mathcal{P}(\omega)|_{fin}$, j an ultrafilter on B and $i = j^* = B - j$ the dual prime ideal. Are there (ω_1, ω_1^*)-gaps between i and j ? The answer is straightforward.

Note that, unlike for $\mathcal{K}$ and $\mathcal{D}$, here this gaps can occur in case the additivity $\underline{b}(i, \subseteq^*)$ is countable and the character $\chi(j)$ in $\beta(\omega)$ $(= \underline{d}(j, \subseteq^*))$ is uncountable.

Observation. For every uniform ultrafilter j on ω there is an (ω_1, ω_1^*)-gap between j^* and j .

Proof. Take $\{A_\alpha : \alpha \in \omega_1\} \subseteq [\omega]^\omega$ and $\{B_\alpha : \alpha \in \omega_1\} \subseteq [\omega]^\omega$ an arbitrary (ω_1, ω_1^*)-gap in $\mathcal{P}(\omega)|_{fin}$ i.e. the following holds: $(\forall \alpha < \beta < \omega_1)(A_\alpha \subseteq^* A_\beta\ \&\ B_\alpha \subseteq^* B_\beta\ \&\ |A_\beta \cap B_\beta| <$ $< \aleph_0)\ \&\ \neg(\exists C \in [\omega]^\omega)(\forall \alpha < \omega_1)(A_\alpha \subseteq^* C\ \&\ |C \cap B_\alpha| < \aleph_0)$.

Take $f : \omega \longrightarrow \omega$ a one-to-one and onto such that $f(B_0) \in j$. Then $\{f(A_\alpha) : \alpha \in \omega_1\} \subseteq j^*$ and $\{f(B_\alpha) : \alpha \in \omega_1\} \subseteq j$ from an (ω_1, ω_1^*)-gap between j^* and j .

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