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DIAGONAL CONDITIONS IN SEQUENTIAL CONVERGENCE

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1. INTRODUCTION

Throughout the paper, each ordinal number is identified with the set of its predecessors, cardinal numbers are initial ordinals, ω denotes the first infinite cardinal, ${}^{\omega}\omega$ denotes the set of all mappings of ω into ω partially ordered "modulo finite" (i.e. $f <^* g$ whenever $f(n) < g(n)$ for all but finitely many $n \in \omega$), each $f \in {}^{\omega}\omega$ is identified with its graph $\{(n, f(n)); n \in \omega\}$, Mon denotes the set of all strictly increasing mappings of ω into ω, a sequence S of points of a set X is considered as a mapping of ω into X and for $s \in$ Mon by $S \circ s$ we denote the composition of s and S; it is the subsequence of S the n-th term of which is $S(s(n))$. If $\langle S_n \rangle$ is a sequence of sequences of points, then to each $f \in {}^{\omega}\omega$ there corresponds a diagonal sequence S_f the n-th term of which is $S_n(f(n))$. Subsequences of diagonals are subdiagonals.

Let X be an infinite set equipped with a sequential convergence structure (e.g., X is a topological space and a sequence $\langle x_n \rangle$ of points converges in X to a point x whenever each neighborhood of x contains x_n for all but finitely many indexes, or a suitable subset $\mathcal{L}$ of $X^{\omega} \times X$ is given and we say than a sequence $\langle x_n \rangle$ $\mathcal{L}$-converges to x whenever $(\langle x_n \rangle, x) \in \mathcal{L}$). We usually assume all four axioms of sequential convergence (FUSH in the Katowice notation or, equivalently, L_i, $i = 0, 1, 2, 3$, in the Prague notation) but in most cases only two of them (viz., (S) For each $x \in X$ the constant sequence $\langle x \rangle$ converges to x; (F) Each subsequence of a convergent sequence converges to the same limit) are relevant. A reader interested in the topological considerations only can assume that X is a sequential space (sequentially open sets are open).

Let V be a sequence of points converging in X to a point v and for each $n \in \omega$ let S_n be a sequence of points converging to the point $V(n)$. Then $(\langle S_n \rangle, V, v)$ is said to be an s-system. Moreover, if V is the constant sequence generated by v (i.e. $V(n) = v$ for all $n \in \omega$), then $(\langle S_n \rangle, v)$ is said to be a p-system. The point v is said to be the vertex of

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$(\langle s_n \rangle, V, v)$, resp. $(\langle s_n \rangle, v)$. By a diagonal condition (abbreviated to DC) for $(\langle s_n \rangle, V, v)$, resp. for $(\langle s_n \rangle, v)$, we understand a condition asserting that for a specified set of diagonals of $\langle s_n \rangle$ specified subdiagonals converge in X to v. If a given DC holds for each s-system, resp. p-system, in X, then we say that X satisfies the DC in question.

As an example consider the following well-known condition: For each p-system $(\langle s_n \rangle, v)$ there exist $f \in {}^{\omega}\omega$ and $s \in$ Mon such that the subdiagonal $S_f \circ s = \langle s_{s(n)}(f(s(n))) \rangle$ converges to v. The condition will be denoted by (PSD); P stands for p-system, SD stands for subdiagonal.

DC's were studied by various authors in connection with: the relationship between a sequential convergence and the induced closure operators ([19], [18], [9], [20]), convergence in products ([19], [18], [13], [2], [7]), topological games ([8], [22], [7]), and analysis ([12]). Some of them were introduced and investigated independently and some of the related results were rediscovered, albeit sometimes in a different setting. In [18], [4], [2] and [21] systems of DC's were considered and, in a sense, all papers cited at the end of the present paper deal with mutual relations between various DC's.

Our aim is to provide a survey on DC's and their applications. First, we present sample results to illustrate the nature and role of DC's. Secondly, we discuss classification schemes for DC's introduced in [18], [4] and [2]. Thirdly, we show that some results obtained earlier under CH hold under weaker (or without any) additional set-theoretical assumptions. Finally, we mention several problems and areas in which the future research on DC's seems to be promising.

It is to be stressed that our survey is far from being complete. Among other results, many elaborated constructions and interesting problems from [2], [12] and [21] are not mentioned here.

We have tried to narrow the gap between the sequential convergence theory (e.g. [18]) and its topological counterpart (e.g. [2], [21]). The idea was to point out results and problems the two sides have in common.

Throughout the literature, the terminology and notation concerning DC's is not uniform (fortunately, it is rather suggestive). To make the cross-reference easier, we usually recall the notation used in the cited papers. For further information on the space ${}^{\omega}\omega$ and its applications to the sequential spaces theory the reader is referred to [6].

2. SAMPLE RESULTS

2.1. "$cl_s = cl_s^2$ "

Let X be a set equipped with a sequential convergence satisfying axioms FS. The sequential closure operator cl_s is defined by $cl_s A = \{x \in X ; x = \lim x_n , x_n \in A , n \in \omega \}$. It is known that the following (subdiagonal) condition

(SD) For each s-system $(\langle S_n \rangle , V, v)$ there exists $f \in {}^{\omega}\omega$ and $s \in \text{Mon}$ such that the subdiagonal $S_f \circ s$ converges to v,

guarantees that $cl_s^2 = cl_s$, i.e. cl_s is idempotent and hence a topological closure. Note that the so-called sequential fan (the quotient of a disjoint sum of countably many convergent sequences with their limit points identified) satisfies $cl_s^2 = cl_s$ but it fails to satisfy (SD). Clearly (SD) implies (PSD). As proved by J. NOVÁK and L. MIŠÍK in [19], the situation for sequential convergence groups is much more convenient.

THEOREM 2.1. ([19], see also [18]) Let X be a commutative group equipped with a compatible sequential convergence (FLUSH). Then $cl_s = cl_s^2$ iff X satisfies (PSD).

REMARK 2.2. It can be readily seen that virtually the same proof works also for noncommutative FLS-convergence groups. P. NYIKOS in [20] proved that if X is a topological group the topology of which is sequential and (PSD) holds, then X is a Fréchet space (i.e. $cl = cl_s$). He also proved that every topological group the topology of which is Fréchet satisfies (PSD). Note that for commutative groups having unique sequential limits both results follow from Theorem 2.1.

2.2. Topological convergences

Let X be a set equipped with a sequential convergence satisfying axioms FS. In [9] among conditions guaranteeing that the convergence of sequences in the topology $cl_s^{\omega_1}$ coincides with the original sequential convergence in X (such convergences are said to be topological) the following two have been considered:

(D) For each s-system $(\langle S_n \rangle , V, v)$ there exists $f \in {}^{\omega}\omega$ such that the diagonal S_f converges to v;

(Y) For each s-system $(\langle S_n \rangle , V, v)$ there exists $f \in {}^{\omega}\omega$ such that for each $g \in {}^{\omega}\omega$, $g > f$, the diagonal S_g converges to v.

A. KAMIŃSKI asked whether in FUSH-convergence spaces (D) implies (Y) (the converse implication holds trivially). Independently, P. KRATOCHVÍL ([11]) and J. BURZYK ([3]) have constructed under CH a FUSH-convergence space, resp. a FLUSH-convergence group, satisfying (D) but not (Y). It will be shown in Section 4 that there exists a FUSH-convergence space satisfying (D) but not (Y) without any additional set-theoretical assumptions.

For a characterization of topological convergences, as well as for further information concerning the relationship between closure operators and sequential convergences, the reader is referred to [10].

2.3. Properties of products

Let X and Y be first-countable topological spaces. Then their product $X \times Y$ is again a first-countable space. Since first-countable spaces are Fréchet, in this case we have two Fréchet spaces the topological product of which is again a Fréchet space. This is not true in general (see Example 2.3). Hence a natural question arises: "Under what conditions the product of two Fréchet spaces is again a Fréchet space?" The problem attracted considerable attention (cf. [2], [13], [23], [7]) and, in fact, it is a special case of the following more general problem: "Under what conditions the closure product of two sequential convergence (FUSH) spaces coincides with their convergence product?" This more general approach to convergence in product is followed in [16], [17], [18]. Most of the DC's considered in our paper have been considered in connection with convergence properties of products (cf. [2]). We start with an illustrative example belonging to folklore (see e. g. [17], [13], [7]). A similar example can be found in [16] on page 22. Note that solving a problem from [13], P. SIMON has constructed in ZFC a compact Fréchet space whose square fails to be Fréchet ([23]).

EXAMPLE 2.3. Let X be the sequential fan (X can be visualized as $(\omega\times\omega)\cup\{v\}$, where points of $\omega\times\omega$ are isolated and a neighborhood base at v is formed by sets $U_f = \{v\}\cup\{(n,m)\in \omega\times\omega ;\ m > f(n),\ f\in {}^{\omega}\omega\}$ and let Y be the space $\omega+1$. Both X and Y are Fréchet spaces with unique sequential limits and hence also FUSH-convergence spaces. Their topological product (the same as closure product; a neighborhood base at (x, y) in the closure product is formed by product of closure neighborhoods of x in X and of y in Y) fails to be a Fréchet space. In their convergence product a sequence $\langle(x_n, y_n)\rangle$ converges to (x, y) iff $\langle x_n\rangle$ converges in X to x and $\langle y_n\rangle$ converges in Y to y. Note that X does not satisfy (PSD) and Y is a first-countable compact space.

THEOREM 2.4. ([18]) Let X and Y be FUSH-convergence spaces. Assume that X satisfies (PSD) and Y satisfies the following condition:

(α) For each $x \in X$ there exists a decreasing sequence $\langle U_n(x)\rangle$ of closure neighborhoods of x such that each one-to-one sequence $\langle x_n\rangle$, $x_n \in U_n(x)$, contains a subsequence converging to x.

Then the closure product and the convergence product of X and Y coincide.

Clearly, each first-countable space satisfies condition (α). Combining Example 2.3 and Theorem 2.4 we get the following characterization of FUSH-convergence spaces satisfying condition (PSD).

COROLLARY 2.5. Let X be a FUSH-convergence space. Then the closure product of X with the closed unit interval I and the convergence product of X and I coincide iff X satisfies (PSD).

LEMMA 2.6. (cf. Lemma 3 in [18]) Let X be a FUSH-convergence space. Then (PSD) is equivalent to the following condition

(δ) If $\langle A_n\rangle$ is a decreasing sequence of subsets of X and $x \in X$ is a point of X such that $x \in cl_s A_n$ for all $n \in \omega$, then there exists a sequence $\langle x_n\rangle$ such that for each $n \in \omega$ we have $x_n \in A_n$ and $\langle x_n\rangle$ converges to x.

REMARK 2.7. Note that if X is a topological space and in (δ) the sequential closure cl_s is replaced by the topological closure cl, then we get the definition of a countably bi-sequential space (sometimes called also a strongly Fréchet space). A. V. ARCHANGELSKIJ proved (Theorem 5.23 in [2]) that if X is a Fréchet space, then X is countably bi-sequential iff X satisfies a certain condition which is equivalent to our (PSD) (ARCHANGELSKIJ considers only one-to-one sequences S_n in p-systems). The same result is in [20] attributed to G. GRUENHAGE. Since, if X is a FUSH-convergence space and $cl_s = cl$, then X is a Fréchet space, this characterization of countably bi-sequential spaces via (PSD) follows from Lemma 2.6. E. MICHAEL proved (Proposition 4.D.5 in [13]) that a space X is countably bi-sequential iff the (topological) product space $X \times I$ is Fréchet. Again, if we restrict ourselves to Fréchet spaces, this characterization of countably bi-sequential spaces follows from Corollary 2.5. This, together with Remark 2.2, shows that in some cases the idempotency of the closure operator in sequential convergence theory is superfluous.

2.4. Analysis

A linear space equipped with a partial order in which each two elements possess a supremum is said to be a Riesz space. As an example, consider a measure space $(X, \mathcal{S}, \mu)$, take the set of all real μ-almost everywhere finite μ-measurable functions on X, equip it with the μ-almost everywhere pointwise partial ordering and identify two functions whenever they are equal μ-almost everywhere. The resulting space $M(X, \mu)$ is a Riesz space. Roughly, the classical Egoroff's theorem states that if in $M(X, \mu)$ a sequence $\langle f_n \rangle$ converges to f pointwise, then this convergence is uniform outside a set of small μ-measure. The theorem has a generalization to abstract Riesz spaces. In this generalization, called in [12] the abstract Egoroff-type theorem, the so-called Egoroff property plays a fundamental role.

Let L be a Riesz space. We say that an element $f \in L$ has the Egoroff property if, given a double sequence $\langle u_{n\,k} \rangle_{n,k \in \omega}$ in L such that for each $n \in \omega$ we have $|f| \geqq u_{n\,k} \downarrow 0$, there exists a sequence $\langle v_m \rangle$ such that $v_m \downarrow 0$ and for every pair (m, n) of natural numbers there exists a natural number $k(m, n)$ such that $v_m \geqq u_{n,\,k(m,\,n)}$ holds. The space L is said to have theEgoroff property if each f in L has the Egoroff property. We say that L has the strong Egoroff property if, given a double sequence $\langle u_{n\,k} \rangle_{n,k \in \omega}$ in L such that for each $n \in \omega$ we have $u_{n\,k} \downarrow 0$, there exists a sequence $\langle v_m \rangle$ such that $v_m \downarrow 0$ and for every pair (m, n) of natural numbers there exists a natural number $k(m, n)$ such that $v_n \geqq u_{n,k(m,n)}$ holds. Evidently, the strong Egoroff property implies the Egoroff property.

The sequence space l_∞ is an example of a space possessing the Egoroff property but not the strong Egoroff property. If μ is σ-finite, then $M(X, \mu)$ has the strong Egoroff property and hence also the Egoroff property.

Let L be a Riesz space. By convergence in L we understand the order convergence, i.e. $\langle f_n \rangle \longrightarrow f$ whenever there exists in L a sequence $\langle p_n \rangle$ such that $p_n \downarrow 0$ and $|f - f_n| \leqq p_n$ holds for all $n \in \omega$. Then the following are equivalent (cf. [12]):

(i) L has the strong Egoroff property;

(ii) L satisfies (D);

(iii) L satisfies (SD) (-called the "diagonal gap property" in [12]).

Further, the strong Egoroff property is equivalent to the fact that the order closure coincides with the sequential closure in L.

An interesting fact about the Riesz spaces theory as presented in [12] is that many fundamental properties of particular Riesz spaces are proved under CH.

2.5. Topological games

Consider the following infinite game introduced by G. GRUENHAGE ([8]). Let X be a topological space and p a point of X. In the n-th step, $n \in \omega$, the first player chooses a neighborhood $\mathcal{N}_n$ of p and the second player chooses a point $p_n \in \mathcal{N}_n$. The first player wins if the sequence $\langle p_n \rangle$ converges to p, otherwise the second player wins. The space X is said to be a <u>w-space</u> if for each $p \in X$ the second player does not have a winning strategy. As shown by P. L. SHARMA ([22]), X is a w-space iff the following condition holds:

(C) If $\langle A_n \rangle$ is a sequence of subsets of X and $p \in \mathrm{cl} A_n$ for each $n \in \omega$, then there exists a sequence $\langle p_n \rangle$ such that $p_n \in A_n$ for each $n \in \omega$ and $\langle p_n \rangle$ converges to p.

As pointed out by G. GRUENHAGE (cf. [20]), w-spaces are precisely Fréchet spaces satisfying the following DC:

(PD) For each p-system $(\langle S_n \rangle, v)$ there exists $f \in {}^{\omega}\omega$ such that the diagonal $S_f = \langle S_n(f(n)) \rangle$ converges to p.

3. CLASSIFICATION OF DC's

In [18], J. NOVÁK introduced the following five conditions for FUSH-convergence spaces (double sequences $\{x_{mn}\}_{m,n=1}^{\infty}$ are considered instead of sequences $\langle S_n \rangle$ of sequences S_n and the term cross-sequence instead of diagonal is used in [18]):

(α) - see Theorem 2.4;

(β) For each p-system $(\langle S_n \rangle, v)$ there exists $f \in {}^{\omega}\omega$ such that for each $g > f$, $g \in {}^{\omega}\omega$, there exists $s \in \mathrm{Mon}$ such that the subdiagonal $S_g \circ s = \langle S_{s(n)}(g(s(n))) \rangle$ converges to v;

(γ) - the same as condition (PSD);

(δ) - see Lemma 2.6;

(ρ) $\equiv \neg(\gamma)$ - included for historical reasons only (cf. [19]).

J. NOVÁK proved that $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma) \Leftrightarrow (\delta)$ and observed that each FUSH-convergence space having a countable (closure) neighborhood base at each point satisfies condition (α). Further, he pointed out that the one-point compactification of a discrete space having uncountable cardinality fulfils (α) without being first-countable, and gave an example of a FUSH-convergence space satisfying (β) but not (α) (Example 3 in [18]). In Section 4 we shall construct (under the assumption that there is a scale in $^{\omega}\omega$) a Fréchet space satisfying (γ) but not (β).

In [4], M. CONTESSA and F. ZANOLIN considered (in our notation) conditions (D), (PD), (SD) and (PSD) for FUSH-convergence spaces. Clearly, (D) $\Rightarrow$ (SD) $\Rightarrow$ (PSD) and (D) $\Rightarrow$ (PD) $\Rightarrow$ (PSD). They asked whether (SD) implies (PD). There are simple examples from which it follows that if the answer is "no", then no two of the four conditions are equivalent. P. MIKUSIŃSKI ([14]) constructed under CH a FUSH-convergence space satisfying (SD) but not (PD). In Section 4 it will be shown that the same result can be obtained without any additional set-theoretical assumption; further, it will be pointed out that for Fréchet spaces conditions (PSD) and (SD) are equivalent.

Yet another system of diagonal conditions was considered by A. V. ARCHANGELSKIJ in [2]. These conditions involve Fréchet (i.e. Fréchet-Urysohn) spaces and p-systems with one-to-one sequences - such systems will be called <u>p^*-systems</u>. The conditions are studied mainly in connection with product spaces. Note that beyond some self-evident cases, he did not investigate the mutual relationship between classes of spaces satisfying his conditions $\langle i - FU \rangle$, $i = 1, \ldots, 5$. In a slightly different notation, four of these conditions appear in the unpublished manuscript [21] by P. NYIKOS, who lectured on this topic at the Colloquium on Topology, held in Eger in 1983. Here we present some of his results and also some of his interesting observations.

Let X be a set equipped with a sequential convergence satisfying the Urysohn axiom

(U) If a sequence S does not converge to $x \in X$, then there exists $s \in \mathrm{Mon}$ such that for each $t \in \mathrm{Mon}$ the subsequence $S \circ s \circ t$ of S does not converge to x.

It is known that if a one-to-one sequence $S = \langle x_n \rangle$ converges in X to a point $x \in X$, then for each one-to-one mapping $s \in {}^{\omega}\omega$ the one-to-one sequence $S \circ s$ also converges to x. Hence in a sequential convergence space satisfying (U), instead of saying that a one-to-one sequence $\langle x_n \rangle$ converges to a point $x \in X$, we can say that the set $S = \{x_n ; n \in \omega\}$ converges to x. From this point of view we shall make no distinction between S

and the corresponding sequence. Now we are ready to recall the conditions studied by A. V. ARCHANGELSKIJ and P. NYIKOS (generalized to sequential convergence spaces; note that only the Urysohn axiom is relevant), and to reformulate them into DC's.

Let X be a FUS-convergence space (e.g. a Fréchet space). Consider the following conditions:

(α_1) For each p^*-system $(\langle S_n \rangle, v)$ there exists a countable set S converging in X to v such that for each $n \in \omega$ we have $|S_n \setminus S| < \omega$ (i.e. all but finitely many elements of the one-to-one sequence S_n belong to S);

(α_2) For each p^*-system $(\langle S_n \rangle, v)$ there exists a countable set S converging in X to v such that for each $n \in \omega$ we have $|S_n \cap S| = \omega$;

(α_3) For each p^*-system $(\langle S_n \rangle, v)$ there exists a countable set S converging in X to v such that for infinitely many $n \in \omega$ we have $|S_n \cap S| = \omega$;

(α_4) For each p^*- system $(\langle S_n \rangle, v)$ there exists a countable set S converging in X to v such that for infinitely $n \in \omega$ we have $S_n \cap S \neq \emptyset$.

Conditions $\langle i - FU \rangle$ are related to (α_j) as follows: $(\alpha_1) \equiv \langle 1 - FU \rangle$, $(\alpha_2) \equiv \langle 5 - FU \rangle$, $(\alpha_3) \equiv \langle 3 - FU \rangle$, $(\alpha_4) \equiv \langle 4 - FU \rangle$. Clearly, for $i = 1, 2, 3$ we have $(\alpha_i) \Rightarrow$ $\Rightarrow (\alpha_{i+1})$. On the other hand, under the assumption that there is no scale in ${}^{\omega}\omega$, P. NYIKOS has constructed a Fréchet space satisfying (α_2) but not (α_1). For $i = 3, 4$ he has examples of Fréchet spaces satisfying (α_i) but not (α_{i-1}) without any additional set-theoretical assumptions.

REMARK 3.1. (see e.g. [14], [21]) Notice that in (α_2) the condition $|S_n \cap S| = \omega$ can be equivalently replaced by $S_n \cap S \neq \emptyset$. Hint. Each p^*-system $(\langle S_n \rangle, v)$ can be replaced by a p^*-system $(\langle S'_n \rangle, v)$ in which $\langle S'_n \rangle$ is obtained by repeating each S_n infinitely many times and each time it appears we leave out from the sequence $\langle S_n(k) \rangle_{k=1}^{\infty}$ larger and larger initial segments. From $S'_n \cap S \neq \emptyset$ we get $|S_n \cap S| = \omega$.

REMARK 3.2. (cf. [21]) Assume that X is a space satisfying condition (α_i) but not (α_j). Then there exists a countable Fréchet space Y all points but one of which are isolated and Y satisfies (α_i) but not (α_j). Indeed, let $(\langle S_n \rangle, v)$ be a p^*-system in which (α_j) fails. Let Y be the underlying set of $(\langle S_n \rangle, v)$. Then Y can be equipped with a sequential convergence in which a sequence S converges to x provided $x = v$ and

S converges in X to v . Clearly, Y has the desired properties. Hence Y is a topological space of the form $\omega \cup \{\mathcal{F}\}$, where $\mathcal{F}$ is a suitable filter on ω and a neighborhood base at $\{\mathcal{F}\}$ is formed by sets $F \cup \{\mathcal{F}\}$, $F \in \mathcal{F}$. Several deep results concerning conditions (α_i) were obtained in [21] along these lines.

REMARK 3.3. For FUSH-convergence spaces, condition (α_1) is equivalent to the following one:

(PY) For each p-system $(\langle S_n \rangle , v)$ there exists $f \in {}^{\omega}\omega$ such that for each $g \in {}^{\omega}\omega$, $g > f$, the diagonal $S_g = \langle S_n(g(n)) \rangle$ converges to v .

Hint. First, observe that "it makes no difference whether we consider p-systems or p^*-systems". Secondly, if $(\langle S_n \rangle , v)$ is a p^*-system and (α_1) holds, then there exists $f \in {}^{\omega}\omega$ such that for each $g \in {}^{\omega}\omega$, $g > f$, the diagonal S_g is a subsequence of S . Thirdly, if $(\langle S_n \rangle , v)$ is a p^*-system and (PY) holds, then for a suitable $f \in {}^{\omega}\omega$ the set $S = \{S_n(k) ;\ u \in \omega$ and $k > f(n)\}$ converges (according to (U)) to v . Clearly $|S_n \setminus S| < \omega$ for each $n \in \omega$.

REMARK 3.4. Similarly, for FUSH-convergence spaces, condition (α_2) is equivalent to (PD) - see Remark 3.1, condition (α_4) is equivalent to (PSD), and condition (α_3) can be transformed into a suitable PDC.

4. IMPROVEMENTS

4.1. (D) versus (Y)

As mentioned in Section 2, condition (D) follows from condition (Y) and, under CH, there are examples of sequential convergence spaces satisfying (D) but not (Y) ([11], [3]); we are going to show that CH can be avoided completely.

P. NYIKOS in [21], under the assumption that there is no scale in ${}^{\omega}\omega$ (recall that a scale in ${}^{\omega}\omega$ is a well-ordered and dominating subset of ${}^{\omega}\omega$), gave an example of a countable Fréchet space, with unique sequential limits, satisfying condition (α_2) but not condition (α_1). Since the space has exactly one nonisolated point, each s-system can be replaced in this space by an "equivalent" p^*-system. Consequently, it follows from Remark 3.1, Remark 3.2 and Remark 3.3 that the space in question satisfies (D) but not (Y).

We are going to construct, under the assumption that there is a scale in ${}^{\omega}\omega$, a Fréchet space X, with unique sequential limits, satisfying (D) but not (Y).

The two examples together imply that, in ZFC, there always exists a Fréchet space with unique sequential limits (i.e. a FUSH-convergence space such that $cl_s = cl_s^2$) which satisfies condition (D) but not (Y).

Our construction is a modification of KRATOCHVÍL's example from [11]. When constructing a sequential convergence space, we have to make a list of convergent sequences in a given set. To kill (Y), we have to construct an s-system $(\langle S_n \rangle, V, v)$ so that for a large (e.g. unbounded) set $\mathcal{F}$ of functions in ${}^{\omega}\omega$, each diagonal $S_f = \langle S_n(f(n)) \rangle$, $f \in \mathcal{F}$, does not converge to the vertex v. A natural candidate for the underlying set of our space X is $((\omega + 1) \times (\omega + 1)) \setminus \{(\omega, n) ; n \in \omega\}$. The convergence in X will be defined in such a way that: the k-th column $C_k = \{(k, n) ; n \in \omega\}$ will converge to (k, ω), $k \in \omega$; the top line $\{(n, \omega) ; n \in \omega\}$ will converge to (ω, ω); for a suitable scale $\mathcal{F}$ in ${}^{\omega}\omega$, each $f = \{(n, f(n)) ; n \in \omega\} \in \mathcal{F}$ will be totally divergent; every one-to-one sequence which is almost disjoint with each C_k, $k \in \omega$, and each $f \in \mathcal{F}$ will converge to (ω, ω).

REMARK 4.1. (cf. [1], [11]) Let X be a set and for each $x \in X$ let $\mathcal{N}_x$ be a system of subsets of X such that $x \in N$ for each $N \in \mathcal{N}_x$. Then a FUS-convergence for X is defined as follows: $\langle x_n \rangle$ converges to x whenever for each $N \in \mathcal{N}_x$ we have $x_n \in N$ for all but finitely many $n \in \omega$. Further, assume that whenever x and y are distinct points of X, then there are almost disjoint sets $M \in \mathcal{N}_x$ and $N \in \mathcal{N}_y$. Then the convergence satisfies also axiom (H) of the uniqueness of sequential limits. Note that if $S = \{x_n \in X ; n \in \omega\}$ is a countable subset of X such that for each $x \in X$ there is a set $N \in \mathcal{N}_x$ such that S and N are almost disjoint, then no subsequence of S converges in X.

Using the just described method, we shall equip X with a FUSH-convergence having the properties mentioned before. It remains to take care of (D). Note that from (D) it will follow that in X we have $cl_s = cl_s^2$ (cf. Section 2.1), and hence X will be a Fréchet space. To verify (D), it suffices to consider only s-systems with vertex (ω, ω). Indeed, since each point $x \in X \setminus \{(\omega, \omega)\}$ has a countable neighborhood base, (D) holds for each s-system with vertex x different from (ω, ω) trivially. Moreover, as we shall see, it suffices to verify that (D) holds only for two basic types of s-systems with vertex (ω, ω) (in fact, one of these two is a p-system).

Our construction is based on two technical lemmas, in which we use the assumption that there is a scale in ${}^{\omega}\omega$. It is known (cf. [5], [6]) that this assumption is independent of axioms of ZFC and it is equivalent to the assumption that the following two cardinal invariants of the space ${}^{\omega}\omega$ are equal: $\underline{b}$ - the least cardinality of an unbounded family of functions in ${}^{\omega}\omega$ and $\underline{d}$ - - the least cardinality of a dominating family of functions in ${}^{\omega}\omega$. Further, under $\underline{b} = \underline{d}$ there is in ${}^{\omega}\omega$ a scale of the cardinality $\underline{b} = \underline{d}$.

Recall that we identify each $f \in {}^{\omega}\omega$ with its graph $\{(n, f(n)) : n \in \omega\} \subset \omega \times \omega$, $\mathrm{dom}(f)$ denotes the domain of f, and $f <^{*} g$ means that $f(n) < g(n)$ for all but finitely many $n \in \omega$.

The first lemma guarantees that if $\langle R_n \rangle$ is a sequence such that each R_n is an infinite subset of the n-th column $C_n = \{(n, k) ; k \in \omega\}$, then there exists a selector $g \in {}^{\omega}\omega$ of the sequence $\langle R_n \rangle$ such that g (i.e. its graph) is almost disjoint with each $f \in \mathcal{F}$ (i.e. with its graph). This will in fact guarantee that in our space X condition (D) holds for each s-system with vertex (ω, ω) each "base" sequence of which is a one-to-one sequence in some column C_n. The selector g will be the required diagonal converging to (ω, ω).

LEMMA 4. 2. Let $\mathcal{F} = \{f_\alpha ; \alpha \in \underline{b} = \underline{d}\}$ be a scale in ${}^{\omega}\omega$. For each $n \in \omega$, let R_n be an infinite subset of ω. Then there exists $g \in {}^{\omega}\omega$ such that:

(i) For each $n \in \omega$ we have $g(n) \in R_n$; and

(ii) For each $f_\alpha \in \mathcal{F}$ the set $g \cap f_\alpha$ is finite.

PROOF. Since each R_n is infinite, by induction it is easy to construct a sequence $\langle g_n \rangle$ in ${}^{\omega}\omega$ such that for each $k \in \omega$ we have $g_n(k) \in R_k$ and an increasing sequence $\langle \alpha_n \rangle$ in $\underline{b} = \underline{d}$ such that for each $n \in \omega$ we have

$$g_n <^{*} f_{\alpha_n} < g_{n+1} .$$

Put $\alpha = \sup \alpha_n$. Then there exists a strictly increasing sequence $\langle k_n \rangle$ in ω such that for each $n \in \omega$ and each $m > k_n$ we have $f_\alpha(m) > g_n(m)$. Define

$$g(m) = g_0(m) \quad \text{if} \quad m \leqq k_1 , \quad \text{and}$$

$$g(m) = g_n(m) \quad \text{if} \quad k_n < m \leqq k_{n+1} .$$

Clearly, $g <^{*} f_\alpha$ and $f_\beta <^{*} g$ whenever $\beta < \alpha$. This completes the proof.

Denote by $\mathcal{P}$ the set $\bigcup\{{}^M\omega ;\ M \in [\omega]^\omega\}$ of all partial functions in ${}^\omega\omega$, i.e. all subsets of $\omega \times \omega$ having the form $g \restriction M = \{(k, g(k)) ;\ k \in M\}$, where g belongs to ${}^\omega\omega$ and M is an infinite subset of ω.

The second lemma asserts that if $\langle g_n \rangle$ is a sequence in $\mathcal{P}$ and each g_n (i.e. its graph) is almost disjoint with each $f \in \mathcal{F}$ (i.e. with its graph), then there is a selector $g \in \mathcal{P}$ of $\langle g_n \rangle$ which is also almost disjoint with each $f \in \mathcal{F}$. This will guarantee that in our space X condition (D) holds for each p-system with vertex (ω, ω) each "base" sequence of which is a one-to-one subdiagonal of the sequence $\langle C_n \rangle$ of columns C_n. Ideas similar to that used in the proof of the lemma were already employed by other authors, see e.g. [5], [21], [15].

LEMMA 4.3. Let $\mathcal{F} = \{f_\alpha ;\ \alpha \in \underline{b} = \underline{d}\}$ be a scale in ${}^\omega\omega$. Let $\langle g_n \rangle$ be a sequence in $\mathcal{P}$ such that for each $\alpha \in \underline{b} = \underline{d}$ and each $n \in \omega$ the set $g_n \cap f_\alpha$ is finite. Then there exists $g \in \mathcal{P}$ such that:

(i) For each $n \in \omega$ we have $g \cap g_n \neq \emptyset$; and

(ii) For each $f_\alpha \in \mathcal{F}$ the set $g \cap f_\alpha$ is finite.

PROOF. For each $n \in \omega$ there exists $\beta_n \in \underline{b} = \underline{d}$ such that $g_n <^* f_{\beta_n}$. Put $\beta = \sup \beta_n$. Then for each $\alpha < \beta$ there exists $d_\alpha \in {}^\omega\omega$ such that for each $n \in \omega$ we have $\mathrm{dom}\,(g_n \cap f_\alpha) \subset d_\alpha(n)$. Since $\beta < \underline{b} = \underline{d}$, there exists $d \in {}^\omega\omega$ such that for each $\alpha < \beta$ we have $d_\alpha <^* d$. Put

$$k_o = \min\{k \in \mathrm{dom}(g_o) ;\ g_o(k) < f_\beta(k) \text{ and } k > d(o)\},$$

and inductively

$$k_{n+1} = \min\{k \in \mathrm{dom}(g_n) ;\ g_n(k) < f_\beta(k) \text{ and } k > d(n+1) + k_n\}.$$

Define

$$g(k_n) = g_n(k_n),\quad n \in \omega .$$

Now, let $\alpha \in \underline{b} = \underline{d}$ be a fixed ordinal. If $\alpha \geqq \beta$, then from $g < f_\beta$ it follows that the set $g \cap f_\alpha$ is finite. If $\alpha < \beta$, then there exists $n_o \in \omega$ such that for each $n \geqq n_o$ we have $d(n) > d_\alpha(n)$. Consequently, since for each $n \geqq n_o$ we have $\mathrm{dom}(g_n \cap f_\alpha) \subset d_\alpha(n) \subset d(n)$ and $k_n > d(n)$, we get $\mathrm{dom}(g \cap f_\alpha) \subset \subset \{k_o, k_1, \ldots, k_{n_o}\}$. This completes the proof.

Now we are ready to describe our space X formally.

EXAMPLE 4.4. Let $\mathcal{F} = \{f_\alpha ;\ \alpha \in \underline{b} = \underline{d}\}$ be a scale in ${}^\omega\omega$. Put $X = ((\omega + 1) \times (\omega + 1)) \setminus \{(\omega, n);\ n \in \omega\}$. For each $x \in X$ define a system $\mathcal{N}_x$ of subsets of X as follows:

(i) For each $m, n \in \omega$, $\mathcal{N}_{(m,n)}$, consists of the singleton $\{(m, n)\}$;

(ii) For each $n \in \omega$, $\mathcal{N}_{(n,\omega)}$ consists of all sets of the form $\{(n,\omega)\} \cup (C_n \setminus F)$, where F is a finite subset of C_n ;

(iii) $\mathcal{N}_{(\omega,\omega)}$ consists of all sets of the form $\{(\omega,\omega)\} \cup (((\omega \times (\omega+1)) \setminus (C_n \cup f_\alpha)) \setminus F)$, where $n \in \omega$, $f_\alpha \in \mathcal{F}$ and F is a finite subset of $(\omega \times (\omega+1))$.

Define a sequential convergence for X as follows:

(iv) $\langle x_n \rangle$ converges to x whenever for each $N \in \mathcal{N}_x$ we have $x_n \in N$ for all but finitely many $n \in \omega$.

It is easy to verify that, according to Remark 4.1, X is a FUSH-convergence space.

We are going to prove that X satisfies condition (D). So, let $(\langle S_n \rangle, V, v)$ be an s-system in X . If $v \neq (\omega,\omega)$, then it follows directly from the construction of X that $(\langle S_n \rangle, V, v)$ satisfies (D) . If $v = (\omega,\omega)$, then a straightforward but tedious calculation shows that $(\langle S_n \rangle, V, (\omega,\omega))$ can be split into a finite number of s-systems, each of which is either trivial or it is of the type described in Lemma 4.2 or Lemma 4.3. Since each of these systems possesses a diagonal converging to (ω,ω) , by gluing them together we get a diagonal of the original s-system $(\langle S_n \rangle, V, (\omega,\omega))$ and, by the Urysohn axiom (U), the diagonal converges in X to (ω,ω) (recall that, assuming (U), if $\langle x_n \rangle$ and $\langle y_n \rangle$ converge to x , then their conjunction $\langle z_n \rangle = \langle x_n \rangle \wedge \langle y_n \rangle$, defined by $z_{2n} = x_n$ and $z_{2n+1} = y_n$, also converges to x and, if $s \in {}^\omega\omega$ is a one-to-one mapping, then also $\langle z_{f(n)} \rangle$ converges to x). Consequently, X satisfies condition (D) .

Observe that from (D) it follows that X is a Fréchet space (cf. Section 2.1).

Finally, since $\mathcal{F}$ is a scale in ${}^\omega\omega$ and each $f_\alpha \in \mathcal{F}$ is a totally divergent sequence, the s-system $(\langle C_n \rangle, \langle (n,\omega) \rangle, (\omega,\omega))$ does not satisfy condition (Y) .

4.2. (β) versus (γ)

As we pointed out in Section 3, the mutual relationship between conditions (β) and (γ) remained in [18] unsolved. We are going to construct, under the assumption that there is a scale in ${}^\omega\omega$, a Fréchet space satisfying (γ) but not (β). The space is, in fact, a modification of Example 4.4 and its properties follow from Lemma 4.2 and Lemma 4.3.

EXAMPLE 4.5. Let $\mathcal{F} = \{f_\alpha ; \alpha \in \underline{b} = \underline{d}\}$ be a scale in ${}^\omega\omega$.

Let Y be a quotient of the space X from Example 4.4, with all points (n, ω) identified to the point (ω, ω). Then Y is the set $(\omega \times \omega) \cup \{v\}$ equipped with a FUSH-convergence determined by systems $\mathcal{N}_y$ of subsets of Y (cf. Remark 4.1) defined as follows:

(i) For each $(m, n) \in \omega \times \omega$, $\mathcal{N}_{(m,n)}$ consists of the singleton $\{(m, n)\}$;

(ii) The system $\mathcal{N}_v$ consists of sets $\{v\} \cup (((\omega \times \omega) \setminus f_\alpha) \setminus F)$, where $f_\alpha \in \mathcal{F}$ and F is a finite subset of $\omega \times \omega$.

Because v is the only nonisolated point in Y, Y is a Fréchet space.

Since $\mathcal{F}$ is a scale in ${}^{\omega}\omega$ and, by (ii), each $f_\alpha \in \mathcal{F}$ is a totally divergent one-to-one sequence, the space Y does not satisfy condition (β) at the p-system $(\langle C_n \rangle, v)$.

To show that Y satisfies $(\gamma) \equiv$ (PSD), it suffices to split each p-system $(\langle S_n \rangle, v)$ into simple p-systems and use Lemma 4.2 and Lemma 4.3 to construct a suitable subdiagonal converging in Y to v. This involves no difficulty and is omitted.

4.3. (SD) versus (PD)

Answering a problem posed in [4], P. MIKUSIŃSKI has constructed in [14] a FUSH-convergence space satisfying (SD) but not (PD). In his construction he has employed the Continuum hypothesis. We are going to show that such space exists in ZFC.

THEOREM 4.6. Let X be a Fréchet space with unique sequential limits (i.e. a FUSH-convergence space in which $cl_s = cl_s^2$). Then conditions (SD) and (PSD) are equivalent in X.

PROOF. Clearly, (SD) implies (PSD). To prove the converse implication, assume that $(\langle S_n \rangle, V, v)$ is an s-system. Then there are three possibilities.

Case I. For some $s \in \mathrm{Mon}$, $V \circ s$ is a constant sequence. Then $V \circ s = \langle v \rangle$ and the existence of a subdiagonal of $(\langle S_n \rangle, V, v)$ converging to v follows from (PSD).

Case II. For some $s \in \mathrm{Mon}$, $V \circ s$ is a one-to-one sequence and for each $n \in \omega$ there exists $k_{s(n)}$ such that $S_{s(n)}(k_{s(n)}) = V(s(n))$. Then $\langle S_{s(n)}(k_{s(n)}) \rangle$ is the required subdiagonal converging to v.

Case III. There exists an s-system $(\langle T_n \rangle, U, v)$ such that U is a one-to-one sequence and $U = V \circ s$ for some $s \in \mathrm{Mon}$, further, each T_n is a one-to-one subsequence of $S_{s(n)}$ and

$v \neq T_n(k)$ for all $k \in \omega$. For $A = \bigcup_{n\in\omega} \bigcup_{k\in\omega} \{T_n(k)\}$ we have $v \in \mathrm{cl}A \setminus A$. Since X is a Fréchet space, there is in A a sequence T converging to v, and, by the uniqueness of sequential limits, T catches at most finitely many points from each T_n, $n \in \omega$. Hence, for some $t \in$ Mon, $T \circ t$ is a subdiagonal of $(\langle S_n \rangle, V, v)$ and it converges to v. This completes the proof.

P. NYIKOS in [21] gave, in ZFC, an example of a Fréchet space with unique sequential limits, satisfying condition (α_4) but not condition (α_3) and hence also not condition (α_2). As shown in Section 3, (α_4) is equivalent to (PSD) and (α_2) to (PD). Hence by Theorem 4.6, the space satisfies (SD) but not (PD).

REMARK 4.7. It might be in place to observe that instead of Theorem 4.6 we could use a similar argument as in Section 4.1. Namely, assuming a set-theoretical assumption A - possibly void, let X be a FUSH-convergence space satisfying (α_i) but not (α_j). Then, by Remark 3.2, there exists a (countable) Fréchet space Y such that: it has unique sequential limits, it has exactly one non-isolated point, it satisfies (α_i) but not (α_j). Since in such space each s-system can be replaced by an "equivalent" p^*-system, we have, under the assumption A, a space satisfying the DC equivalent to (α_i) but not the DC equivalent to (α_j).

5. CONCLUDING REMARKS

At the end of our paper we would like to mention several areas in which there are interesting unsolved problems concerning DC's.

1. Gaps in the classification of DC's. We have mentioned several examples (showing that one DC does not imply another one) constructed under additional set-theoretical assumptions. It would be nice to have the corresponding absolute results. Here we recall the problem from [21] whether there is in ZFC a Fréchet space satisfying (α_2) but not (α_1). Also the exact position of condition $(\mathfrak{p})$ in the hierarchy of other DC's seems to be an interesting problem.

2. DC's in sequential convergence groups. Even though there are some isolated results concerning the mutual relationship between various DC's in FLUSH-convergence groups (cf. [18], [12], [21], [3]) the situation is far from being satisfactory. Problems in this area seem to be particularly hard.

3. Analysis. In [12] there are several interesting theorems and examples using CH. Is it possible to avoid this assumption?

4. DC's in special classes of spaces. As pointed out in [2] and [21], very little is known about special classes of spaces, such as compact, regular, sequentially compact, homogeneous, etc., in which a particular DC holds.

It is our feeling that there are also other areas in mathematics where DC's play an important role. But that is another story.

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