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# THE SPACE ${}^{\omega}\omega$ IN SEQUENTIAL CONVERGENCE

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ABSTRACT. We survey cardinal invariants and combinatorial phenomena connected with the space ${}^{\omega}\omega$ of all mappings of $\omega$ into $\omega$ equipped with the partial order "modulo finite" and the algebra $\mathcal{P}(\omega)/_{fin}$ of all subsets of $\omega$ "modurlo finite". We discuss recent constructions of peculiar sequential spaces in which infinite combinatorics has been exploited.

## 1. INTRODUCTION

The space ${}^{\omega}\omega$ of all mappings of natural numbers into natural numbers equipped with the partial order "modulo finite" (i.e., for $f, g \in {}^{\omega}\omega$ we put $f <^* g$ whenever $f(n) < g(n)$ for all but finitely many $n \in \omega$ ) and the algebra $\mathcal{P}(\omega)/_{fin}$ are extremely useful tools for specialists in sequential convergence. As it is known for long, some of the properties of ${}^{\omega}\omega$ and $\mathcal{P}(\omega)/_{fin}$ depend on the particular model of set theory we are working with. An excellent account of the properties of ${}^{\omega}\omega$ and $\mathcal{P}(\omega)/_{fin}$ with applications to topology was written by E. K. van DOUWEN ([11]). In our paper we would like to stress the importance of some combinatorial ideas in sequential convergence. We make some remarks on constructions of peculiar sequential spaces based on ${}^{\omega}\omega$, survey cardinal invariants and combinatorial phenomena connected with ${}^{\omega}\omega$ and $\mathcal{P}(\omega)/_{fin}$ and present some old and new results around ${}^{\omega}\omega$ and $\mathcal{P}(\omega)/_{fin}$.

### 1.1. Definitions, notation, basic facts.

For the reader s convenience, in this section we recall the basic terminology and notation. Ordinals are von Neumann ordinal numbers (i.e., the sets of all predecessors), cardinals are initial ordinals, $\omega$ is the set of all natural numbers - the least infinite ordinal (cardinal) number.

The set theory we are working with is ZFC (the Zermelo-Fraenkel set theory with the axiom of choice); CH denotes the Continuum hypothesis $2^{\omega} = \omega_1$. Our standard reference book on set theory is [23].

Functions in ${}^{\omega}\omega$ are identified with their graphs in $\omega \times \omega$. A set $\mathcal{B} \subset {}^{\omega}\omega$ is said to be unbounded if $(\forall g \in {}^{\omega}\omega)\ (\exists f \in \mathcal{B}\ )$

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$(f \not\leq^* g)$ . Denote by $\underline{b}$ the minimal size of an unbounded set in ${}^{\omega}\omega$ . It is a folklore that there is a family $\mathcal{F} = \{f_\alpha : \alpha \in \underline{b}\} \subset {}^{\omega}\omega$ such that: $\mathcal{F}$ is unbounded, $\alpha < \beta$ implies $f_\alpha <^* f_\beta$ (i.e., $\mathcal{F}$ is well-ordered), each $f_\alpha$ is increasing (note that if M is an infinite subset of $\omega$ , then $\mathcal{F}$ restricted to M has the same properties in ${}^{M}\omega$ , the space of all mappings of M into $\omega$ , with respect to the order inherited from ${}^{\omega}\omega$) . A set $\mathcal{D} \subset {}^{\omega}\omega$ is said to be <u>dominating</u> if $(\forall g \in {}^{\omega}\omega)(\exists f \in \mathcal{D})(g <^* f)$ . Denote by $\underline{d}$ the minimal size of a dominating family of functions in ${}^{\omega}\omega$ . A set $\mathcal{S} \subset {}^{\omega}\omega$ is said to be a <u>scale</u> if it is dominating and well-ordered under $<^*$ . The existence of a scale in ${}^{\omega}\omega$ is independent of the axioms of ZFC, it follows from CH , and it is equivalent to the condition $\underline{b} = \underline{d}$ .

Let X be an infinite set. Denote by $[X]^{\omega}$ , resp. by $[X]^{<\omega}$, the set of all countable, resp. finite, subsets of X . For A, B $\in [X]^{\omega}$ , we say that they are <u>almost disjoint</u> if $A \cap B$ is finite, and we write $A \subset^* B$ if $A \setminus B$ is finite. A set $\mathcal{A} \subset [X]^{\omega}$ is said to be an <u>almost disjoint family</u>, abbr. ADF, if $(\forall A, B \in \mathcal{A}, A \neq B)$ $(A \cap B$ is finite), resp. a <u>maximal almost disjoint family</u>, abbr. MADF, if $\mathcal{A}$ is an ADF and it is not contained properly in any other ADF in $[X]^{\omega}$ .

Consider the Boolean algebra $\mathcal{P}(\omega)$ of all subsets of $\omega$ . Then $\mathcal{P}(\omega)/_{fin}$ is the factor algebra of $\mathcal{P}(\omega)$ modulo the ideal of all finite subsets of $\omega$ ; instead of a factor class $[M]$, $M \subset \omega$ , we sometimes work with its representative M . We recall the Stone duality between $\omega^* = \beta\omega \setminus \omega$ (the remainder of the Čech-Stone compactification $\beta\omega$ of the discrete space $\omega$ ) and $\mathcal{P}(\omega)/_{fin}$ . Points of $\omega^*$ correspond to free ultrafilters on $\omega$ , clopen sets to subsets of $\omega$ , open sets to ideals, closed sets to filters, regular open dense sets to maximal almost disjoint families, nowhere dense sets to free filters (cf. [40], [12]).

We often work with spaces of the type $N \cup \mathcal{N}$ (denoted also by $\Psi(\omega, \mathcal{N})$), where N is a countable infinite set and $\mathcal{N} \subset [N]^{\omega}$ is an AD family; points in N are isolated and a neighborhood base at $A \in \mathcal{N}$ is formed by sets $\{A\} \cup (A \setminus F)$ , F is a finite subset of A .

## 2. MOTIVATION

Mathematicians usually work in ZFC but sometimes use certain additional set-theoretical assumptions (CH , $\underline{b} = \underline{d}$ , etc.) and their consequences. Of course, the weaker assumption - the better. To illustrate the nature of ${}^{\omega}\omega$ and its role in sequential convergence, we start with three closely related examples of Fréchet spaces each of which is sequentially regular but fails to be regular. Recall that a space $X$ is said to be <u>sequentially regular</u> if the convergence of sequences in $X$ is projectively generated by the set $C(X)$ of all continuous functions on $X$, i.e., $x_n \longrightarrow x$ in $X$ iff for each $f \in C(X)$ we have $f(x_n) \longrightarrow f(x)$. Note that for sequential convergence spaces (cf. [33]) the sequential regularity plays an analogous role as the complete regularity does for topological spaces.

EXAMPLE 1. Assuming CH, let $\mathcal{F} = \{f_\alpha : \alpha \in \omega_1\}$ be a scale in ${}^\omega\omega$. Consider the set $Y_a = (\omega \times (\omega + 1)) \cup \omega_1$ equipped with the following topology: all points in $\omega \times \omega$ are isolated; for each $\alpha \in \omega_1$, a neighborhood base at $\alpha$ is formed by sets $\{\alpha\} \cup (f_\alpha \setminus F)$, where $F$ is a finite subset of $f_\alpha = \{(n, f_\alpha(n)) : n \in \omega\}$; for each $n \in \omega$, a neighborhood base at $(n, \omega)$ is formed by sets $\{(n, \omega)\} \cup (C_n \setminus F)$, where $C_n = \{(n, m) : m \in \omega\}$ and $F$ is a finite subset of $C_n$. Clearly, $Y_a$ is a space of the type $\Psi(\omega, \mathcal{N})$. Finally, let $X_a$ be the quotient space obtained from $Y_a$ by identifying points $(n, \omega)$ to one point $p$. Then $X_a$ is a Fréchet space, homeomorphic to the space $L_9$ in [33]. Using the fact that $\mathcal{F}$ is well-ordered, it can be shown that $X_a$ is sequentially regular. (Hint. Assume $x_n \not\longrightarrow x$. We have to find a continuous function $\varphi$ such that $\varphi(x_n) \not\longrightarrow \varphi(x)$. The case $x \neq p$ is trivial. Let $x = p$. For each $\alpha \in \omega_1$, define a function $\varphi_\alpha$ on $X_a$ as follows: $\varphi_\alpha((n, m)) = 0$ whenever $m < f_\alpha(n)$, $\varphi_\alpha(\beta) = 0$ whenever $\beta < \alpha$, and $\varphi_\alpha(y) = 1$ otherwise. For each $\alpha \in \omega_1$, $\varphi_\alpha(p) = 1$ and, since $\mathcal{F}$ is well-ordered, $\varphi_\alpha$ is a continuous function. Clearly, there exists $\alpha \in \omega_1$ such that $\varphi_\alpha(x_n) = 0$ for infinitely many $n \in \omega$.) On the other hand, using the fact that $\mathcal{F}$ is dominating, it can be shown that $X_a$ fails to be regular. (Hint. Sets $O_f = \{p\} \cup \{(m, n) \in \omega \times \omega : m > f(n)\}$, $f \in {}^\omega\omega$, form a neighborhood base at $p$. Since $\mathcal{F}$ is a dominating family, $(\forall f \in {}^\omega\omega)(\exists \alpha \in \omega_1)(f <^* f_\alpha)$. Further, $f <^* f_\alpha$ implies that $f_\alpha \setminus O_f$ is a finite set, and hence $\alpha \in \operatorname{cl} O_f$. Consequently, the point $p$ and the closed set $\omega_1$ cannot be separated by open sets.)

Note that CH in Example 1 can be trivially weakened to the assumption that there is a scale in ${}^\omega\omega$. Further, as pointed out in [17], Example 1 can be modified so that even this assumption can be completely avoided, viz. the scale $\mathcal{F}$ can be replaced by a well-ordered unbounded family $\mathcal{F}$ in ${}^\omega\omega$ (such family always exists in ZFC).

EXAMPLE 2. Let $\{f_\alpha \in {}^\omega\omega : \alpha \in \underline{b}\}$ be a well-ordered ($f_\alpha <^* f_\beta$ iff $\alpha < \beta$ ) unbounded family in ${}^\omega\omega$ . For $Y_b = (\omega \times (\omega + 1)) \cup \underline{b}$ define a topology in the same way as for $Y_a$ in Example 1 and, identifying all points $(n, \omega)$ to one point $p$ , consider $X_b = (\omega \times \omega) \cup \{p\} \cup \underline{b}$ equipped with the corresponding quotient topology. Again, using virtually the same argument as in Example 1, it can be shown that $X_b$ is a sequentially regular Fréchet space which fails to be regular (for each $0_f$ there exists $\alpha \in \underline{b}$ such that $0_f \cap f_\alpha$ is an infinite set, and hence $\alpha \in \mathrm{cl}\ 0_f$ ).

Spaces $X_a$ and $X_b$ are uncountable. As shown in [16], even a countable Fréchet space can be sequentially regular but not regular.

EXAMPLE 3. Let $\mathcal{A}$ be an ADF on $\omega$ such that the cardinality of $\mathcal{A}$ is $2^\omega$ (such $\mathcal{A}$ always exists in ZFC). Since the cardinality of ${}^\omega\omega$ is $2^\omega$, there exist one-to-one correspondences $\mathcal{A} = \{A_\alpha : \alpha \in 2^\omega\}$ between $\mathcal{A}$ and $2^\omega$ and ${}^\omega\omega = \{f_\alpha : \alpha \in 2^\omega\}$ between ${}^\omega\omega$ and $2^\omega$ . Consider the set $Y_c = (\omega \times (\omega + 1)) \cup 2^\omega$ equipped with the following topology: all points in $\omega \times \omega$ are isolated; for each $n \in \omega$ , a neighborhood base at $(n, \omega)$ is formed by sets $\{(n, \omega)\} \cup (C_n \setminus F)$ , where $C_n = \{(n, m) : m \in \omega\}$ and $F$ is a finite subset of $C_n$ ; for each $\alpha \in 2^\omega$ , a neighborhood base at $\alpha$ is formed by sets $\{\alpha\} \cup (f_\alpha \upharpoonright A_\alpha \setminus F)$ , where $F$ is a finite subset of $f_\alpha \upharpoonright A_\alpha$ , the restriction of $f_\alpha$ to $A_\alpha$ . Again, $Y_c$ is a space of the type $\Psi(\omega, \mathcal{N})$ . Let $X_c$ be the quotient space obtained from $Y_c$ by identifying all points $\alpha \in 2^\omega$ to one point $p$ . It is not so hard to verify that $X_c$ has the desired properties.

All three examples are quotients of spaces of the type $N \cup \mathcal{N}$ ($\equiv \Psi(\omega, \mathcal{N})$) with $N = \omega \times \omega$ . The properties of $\mathcal{N}$ , and hence of the whole space $N \cup \mathcal{N}$ , depend on whether or not there is in ${}^\omega\omega$ a certain family of functions. This, however, may depend on additional set-theoretical assumptions.

## 3. REMARKS ON CARDINAL INVARIANTS

In this section we recall the definitions of certain cardinal invariants and make some remarks concerning their applications. A mnemonic notation of these invariants (they are usually denoted by lower case German letters) is largerly due to E. van DOUWEN; $\underline{h}$ is due to P. NYIKOS and $\underline{n}$ is due to B. BALCAR, J. PELANT and P. SIMON.

Denote by $\underline{b}$ the minimal size of an unbounded family of functions in ${}^{\omega}\omega$, i.e.,

$$\underline{b} = \min\{|\mathcal{B}| : \mathcal{B} \subset {}^{\omega}\omega \ \& \ (\forall g \in {}^{\omega}\omega)(\exists f \in \mathcal{B})(f \not<^{*} g)\};$$

denote by $\underline{d}$ the minimal size of a dominating family of functions in ${}^{\omega}\omega$, i.e.,

$$\underline{d} = \min\{|\mathcal{D}| : \mathcal{D} \subset {}^{\omega}\omega \ \& \ (\forall g \in {}^{\omega}\omega)(\exists f \in \mathcal{D})(g <^{*} f)\};$$

denote by $\underline{p}$ the minimal size of a free filter with the pseudo-intersection property or, equivalently, the minimal size of a neighborhood base of a nowhere dense subset of $\omega^{*}$, i.e.,

$$\underline{p} = \min\{|F| : F \subset \mathcal{P}(\omega)/_{\mathrm{fin}} \ \& \ F \text{ is centred } \& \ \bigwedge F = 0\} =$$
$$= \min\{|\mathcal{F}| : \mathcal{F} \subset [\omega]^{\omega} \ \& \ (\forall \mathcal{G} \in [\mathcal{F}]^{<\omega})(\bigcap \mathcal{G} \text{ is infinite}) \ \& \ (\forall X \in [\omega]^{\omega})(\exists Y \in \mathcal{F})(X \setminus Y \text{ is infinite})\};$$

denote by $\underline{t}$ the minimal size of a nowhere dense tower of $\omega^{*}$, i.e.,

$$\underline{t} = \min\{|T| : T \subset \mathcal{P}(\omega)/_{\mathrm{fin}} \ \& \ (T, <) \text{ is well-ordered } \& \ \& \ \bigwedge T = 0\} =$$
$$= \min\{|\mathcal{T}| : \mathcal{T} \subset [\omega]^{\omega} \ \& \ (\mathcal{T}, \subset^{*}) \text{ is well ordered } \& \ \& \ (\forall X \in [\omega]^{\omega})(\exists Y \in \mathcal{T})(X \setminus Y \text{ is infinite})\};$$

denote by $\underline{h}$ the minimal size of the height of a base matrix (a tree-like $\pi$-base), i.e.,

$$\underline{h} = \min\{|\mathcal{F}| : (\forall F \in \mathcal{F})(F \text{ is a nowhere dense subset of } \omega^{*}) \ \& \ \& \ (\bigcup \mathcal{F} \text{ is dense in } \omega^{*})\} =$$
$$= \sup\{\varkappa : (\forall \mu \in \varkappa)(\forall \text{ system } \mathbb{P} = \{\mathcal{P}_{\alpha} : \alpha \in \mu\} \text{ of MAD families on } \omega)(\exists \mathcal{P} \subset [\omega]^{\omega})(\mathcal{P} \text{ is a MAD family } \& \ \mathcal{P} \text{ refines } \mathbb{P})\},$$

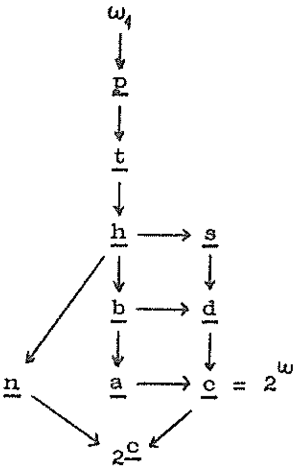
where $\mathcal{P}$ refines $\mathbb{P}$ means that $(\forall \alpha \in \mu)(\forall X \in \mathcal{P})(\exists ! Y \in \mathcal{P}_{\alpha})(X \subset^{*} Y)$;

denote by $\underline{n}$ the so-called Novák number (also called the Baire number) of $\omega^{*}$, i.e.,

$$n = \min\{|\mathcal{F}| : (\forall F \in \mathcal{F})(F \text{ is a nowhere dense subset of } \omega^{*}) \ \& \ \& \ \bigcup \mathcal{F} = \omega^{*}\} =$$
$$= \sup\{\varkappa : (\forall \mu \in \varkappa)(\forall \text{ system } \mathbb{P} = \{\mathcal{P}_{\alpha} : \alpha \in \mu\} \text{ of MAD families on } \omega)(\exists \mathcal{U} \in \omega^{*})(\forall \alpha \in \mu)(\exists P \in \mathcal{P}_{\alpha})(P \in \mathcal{U})\}.$$

Further, we just mention the following two cardinal invariants: $\underline{a}$ denotes the minimal size of a MAD family on $\omega$ and $\underline{s}$ denotes the minimal size of a splitting family in $\mathcal{P}(\omega)/_{\mathrm{fin}}$.

In ZFC, the following inequalities can be proved (in the diagram $\alpha \longrightarrow \beta$ means that $\alpha \leqq \beta$):



Note that B. BALCAR, J. PELANT and P. SIMON proved ([2]) that $\underline{h} \leqq \underline{b}$ ; R. C. SOLOMON proved ([42]) that $\underline{b} \leqq \underline{a}$ ; as communicated to us, P. SIMON and independently P. NYIKOS proved $\underline{s} \leqq \underline{d}$ . The remaining inequalities belong to folklore. The exact relationship between $\underline{a}$ and $\underline{d}$ is not known (there are models in which $\underline{a} = \underline{d}$ or $\underline{a} < \underline{d}$ , but we do not know whether there is a model in which $\underline{d} < \underline{a}$ ) . Also we do not know whether there is a model in which $p < t$ (in all known models we have $p = t$ ) . It follows from the literature ([27], [23], [38], [39]) that, except the two just mentioned unsolved problems, nothing can be added to the above diagram and it is consistent with ZFC that all other inequalities in the above diagram can be sharp.

Next, we present some results from the set-theoretical topology. In doing so, we follow two principles: - using some additional set-theoretical assumptions concerning cardinal invariants, it is possible to prove nonabsolute results; - using some combinatorial objects of nonspecified cardinalities (existing in ZFC), it is possible to get absolute results. Also, we would like to stress the usefulness of combinatorial methods in sequential topology, sequential convergence spaces, general topology, analysis, functional analysis and probability.

S. MRÓWKA ([31]) constructed a MAD family $\mathcal{N}$ such that $\beta(N \cup \mathcal{N})$ is the one-point compactification of $N \cup \mathcal{N}$ . P. SIMON ([41]) constructed (in ZFC) a compact Fréchet space the square of which is not a Fréchet space. A. J. BERNER ([43]), under $\underline{a} = 2^{\omega}$, constructed a noncompact space X such that $\beta X$ is Fréchet. J. NOVÁK ([32]) constructed a regular sequential space on which each (real-valued) continuous function is constant. M. HUŠEK ([15]) constructed a space $N \cup \mathcal{N}$ having different sequential and $\{0, 1\}$-sequential envelopes. Solving a problem from [15], L. MI-

ŠÍK, Jr. ([30]) constructed under $\underline{h} = 2^{\omega}$ a $\{0, 1\}$-sequentially regular space $N \cup \mathcal{N}$ which is sequentially complete but not $\{0, 1\}$-sequentially complete. G. GRUENHAGE ([20]) proved that the product $S_{\omega} \times S_{\varkappa}$ is a sequential space iff $\underline{b} \geq \varkappa^{+}$, where $S_{\alpha}$ is the quotient space of a disjoint union of $\alpha$-many convergent sequences with their limits identified; from this it follows, e.g., that the question "Is $S_{\omega} \times S_{\omega_1}$ a sequential space?" is undecidable in ZFC. A. BLASZCZYK and A. SZYMAŇSKI ([5]) proved that, under CH, for each $p \in \omega^{*}$ the space $\omega^{*} \setminus \{p\}$ is not normal. V. V. FEDORČUK ([13]) proved that, under $\underline{s} = \omega_1$ & $2^{\omega_1} = 2^{\omega}$, there is a compact space of the cardinality $2^{\omega}$ in which no nontrivial sequence converges. B. BALCAR and P. VOJTÁŠ ([3]) proved that in ZFC each point in $\omega^{*}$ is a $2^{\omega}$-point. Using MAD families, M. CONTESSA and F. ZANOLIN ([9]) solved a problem of J. Novák concerning the classification of sequential convergence spaces. A. KAMIŃSKI in a series of papers studied conditional probabilities (cf. [25]) and, under CH, he recently obtained some interesting results in this area. Under CH, P. KRATOCHVÍL ([26]) and J. BURZYK ([7]) constructed a convergence space, resp. a convergence group satisfying condition D but not condition Y; more information about related problems and results can be find in a paper by the present authors in these Proceedings. V. I. MALYHIN and N. N. HOLŠČEVNIKOVA ([22], [29]) proved that, under $\underline{t} = 2^{\omega}$, each dominating family of convergent series (in a natural partial ordering) has the cardinality $2^{\omega}$. Nice applications of combinatorial methods in sequential spaces and their generalizations can be found in the classical paper [1] by A. V. ARHANGEĽSKII and in a recent paper by Z. FROLÍK, R. ISLER and G. TIRONI ([19]).

In the last decade several interesting books and survey articles on cardinal invariants and combinatorial methods have been published, e.g., by M. E. RUDIN ([35]), W. W. COMFORT ([8]), I. JUHÁSZ ([24]), G. M. REED ([34]), or are to be published, e.g., the Handbook of set-theoretic topology, edited by J. E. VAUGHAN and K. KUNEN. The classical textbook [37] by W. SIERPIŃSKI on infinite combinatorics is still motivating.

## 4. NEW RESULTS

In this section we discuss the role and nature of $\underline{h}$ and $\underline{n}$ - two important cardinal invariants of $\mathcal{P}(\omega)/_{\mathrm{fin}}$. Answering a question asked by E. K. van DOUWEN in [11], we illustrate how these invariants can be used in proofs by transfinite induction and in constructions of certain spaces.

Let X be a topological space. Recall that X said to be <u>sequentially compact</u> if $(\forall A \in [X]^{\omega})(\exists B \in [A]^{\omega})(\exists x \in X)$ (B converges to x), i.e., each sequence has a convergent subsequence, and X is said to be <u>countably compact</u> if $(\forall A \in [X]^{\omega})$ $(\exists x \in X)$ (A clusters at x), i.e., each infinite set has a cluster point. Following [11] we define

$\mu = \min\{\varkappa$ : some product of $\varkappa$-many sequentially compact spaces is not sequentially compact$\}$.

C. T. SCARBOROUGH and A. H. STONE ([36]) proved that $\mu \geqq \omega_1$ and S. H. HECHLER ([21]) observed that, in fact, their proof yields $\mu \geqq \underline{t}$. D. BOOTH ([63]) showed that $\mu \leqq \underline{s}$. J. PELANT and P. SIMON independently showed that $\mu \leqq \underline{h}$. Question 6.10 in [11] reads as follows: "Can $\mu$ be expressed as a set-theoretically defined cardinal number?" The answer is provided by the next theorem. P. SIMON informed us that the same result has been obtained by P. NYIKOS, J. PELANT and P. SIMON.

THEOREM. $\mu = \underline{h}$.

PROOF. Since we already know that $\mu \leqq h$, it suffices to prove $\mu \geqq h$. In order to make it more instructive, we break the proof into several steps and add some remarks about the ideas used and the strategy we follow:

1. Remark. Assume that $X_\alpha$, $\alpha \in \varkappa$, are sequentially compact spaces. Following C. T. SCARBOROUGH and A. H. STONE ([36]) and S. H. HECHLER ([21]), if $\{y_n : n \in \omega\}$ is a sequence in the product space $\prod X_\alpha$, then we construct inductively a tower $\{A_\alpha : \alpha \in \varkappa\}$ in $\omega$ in such a way that $\{\mathrm{proj}_\alpha(y_n) : n \in A_\alpha\}$ is a convergent sequence in the space $X_\alpha$. Now, provided $\varkappa < \underline{t}$, the properties of $\underline{t}$ guarantee the existence of a subsequence $\{y_n : n \in A\}$ of $\{y_n : n \in \omega\}$ which converges in $\prod X_\alpha$, i.e., for each $\alpha \in \varkappa$, its projection into $X_\alpha$ converges to some point in $X_\alpha$. The idea of our proof is to replace $\underline{t}$ by $\underline{h}$. For, the properties of $\underline{h}$ will permit us to work simultaneously with a large numbers of "towers of convergent subsequences" (instead of with just one as before) and the induction "will be alive with probability 1" for $\varkappa < \underline{h}$.

2. Lemma. Let X be a sequentially compact space and let $\{y_n : n \in \omega\}$ be a sequence in X. Let $\mathcal{B}$ be a MAD family on $\omega$. Then there is a MAD family $\mathcal{A}$ on $\omega$ such that:

(i) $(\forall A \in \mathcal{A})(\{y_n : n \in A\}$ converges in X);

(ii) $\mathcal{A}$ refines $\mathcal{B}$.

(Hint. An AD family which is maximal with respect to convergence is also maximal with respect to the inclusion; property (ii) is trivial.)

3. Assume $\mathcal{H} < \underline{h}$. Let $X_\alpha$, $\alpha \in \mathcal{H}$, be sequentially compact spaces and let $\prod X_\alpha$ be their product. Let $\{y_n : n \in \omega\}$ be a sequence in $\prod X_\alpha$. Then there are two families $\{\mathcal{A}_\alpha : \alpha \in \mathcal{H}\}$ and $\{x_A^\alpha : \alpha \in \mathcal{H}$ & $A \in \mathcal{A}_\alpha\}$ such that:

(iii) $(\forall \alpha \in \mathcal{H})(\mathcal{A}_\alpha$ is a MAD family on $\omega)$;

(iv) $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ whenever $\alpha, \beta \in \mathcal{H}$ & $\alpha < \beta$;

(v) $(\forall \alpha \in \mathcal{H})(\forall A \in \mathcal{A}_\alpha)(x_A^\alpha \in X_\alpha$ & $\{\mathrm{proj}_\alpha(y_n) : n \in A\}$ converges in $X_\alpha$ to $x_A^\alpha)$.

The construction of $\{\mathcal{A}_\alpha : \alpha \in \mathcal{H}\}$ and $\{x_A^\alpha : \alpha \in \mathcal{H}$ & $A \in \mathcal{A}_\alpha\}$ can be carried out by induction on $\alpha$. If $\alpha$ is isolated, we simply use our Lemma. If $\alpha$ is a limit ordinal, we moreover use some refinement of the previous MAD families. The calculations are straightforward and are omitted. Now, since $\mathcal{H} < \underline{h}$, there is a MAD family $\mathcal{A}$ on $\omega$ such that $\mathcal{A}$ refines the matrix $\{\mathcal{A}_\alpha : \alpha \in \mathcal{H}\}$. Choose an arbitrary $A \in \mathcal{A}$. Then there is a unique family $\{A_\alpha : \alpha \in \mathcal{H}\}$ such that $(\forall \alpha \in \mathcal{H})$ $(A_\alpha \in \mathcal{A}_\alpha$ & $A \subset^* A_\alpha)$. In the product space $\prod X_\alpha$, denote by $y$ the point the $\alpha$-th projection $\mathrm{proj}_\alpha(y)$ of which is $x_A^\alpha$, $\alpha \in \mathcal{H}$. From the construction of the family $\{A_\alpha : \alpha \in \mathcal{H}\}$ it follows readily that the subsequence $\{y_n : n \in A\}$, of the original sequence $\{y_n : n \in \omega\}$, converges to $y$. This completes the proof.

REMARK. Observe that if (in the same notation as in the above proof) $\mathcal{H} < \underline{n}$, then there is an ultrafilter $\mathcal{U}$ on $\omega$ such that $(\forall \alpha \in \mathcal{H})(\exists A_\alpha \in \mathcal{A}_\alpha)(A_\alpha \in \mathcal{U})$. Then the point $y$ (defined by $\mathrm{proj}_\alpha(y) = x_A^\alpha$, $\alpha \in \mathcal{H}$) is an accumulation point of the original sequence $\{y_n : n \in \omega\}$.

COROLLARY. Every product of less than $\underline{n}$ sequentially compact spaces is countably compact.

Observe that it is not known whether in ZFC there is a family $\{X_\alpha, \alpha \in A\}$ of sequentially compact spaces such that the product space $\prod X_\alpha$ is not countably compact (cf. [11]).

It is known (cf. [10]) that there are maximal towers having cardinalities different from $\underline{t}$. Towers can be used, e.g., in constructions similar to that given by S. P. FRANKLIN and M. RAJAGOPALAN ([14]), cf. Example 7.1 in [11].

Concerning numbers $\underline{h}$ and $\underline{n}$, observe that they were used in [18] to construct a completely regular space X in which no nontrivial sequence converges but in the Čech-Stone compactification $\beta X$ there is a nontrivial convergent sequence lying together with its limit in $\beta X \setminus X$ .

## 5. CONCLUDING REMARKS

We have tried to point out some aspects of set theory which are connected with sequential convergence. On one side, they provide tools for constructions and help us to prove theorems but, on the other side, they sometimes fix limits beyond which our attempts to prove some assertions necessarily fail, e.g., if we are trying to prove in ZFC something which is, in fact, independent of ZFC.

We conclude with few remarks concerning the usage of cardinal invariants:

$\underline{b}$ and $\underline{d}$ are useful when constructing spaces of the type $N \cup \mathcal{N}$ for $N = \omega \times \omega$ and $\mathcal{F} \subset {}^{\omega}\omega$ , $\mathcal{F} \subset \mathcal{N}$ , is indexed by $\underline{b}$ or $\underline{d}$ , see e.g. Example 2 ;

t - it helps us to keep the transfinite induction alive, see e.g. [36], [21];

$\underline{h}$ and $\underline{n}$ help us to keep the transfinite induction alive simultaneously "with probability 1", see e.g. Theorem, or they are useful when constructing spaces, see e.g. [17].

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