CONVERGENT SEQUENCES IN βX

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ABSTRACT. Our aim is to construct a completely regular Hausdorff topological space X in which no nontrivial sequence converges and in its Čech-Stone compactification βX there is a nontrivial convergent sequence. We show that all three possibilities occur: (IN-OUT) the sequence is in X and its limit point is in $\beta X - X$, (OUT-IN) the sequence is in $\beta X - X$ and its limit point is in X and, finally, (OUT-OUT) both the sequence and its limit point are in $\beta X - X$. We discuss the minimal cardinality of the spaces in question.

Let X be a completely regular Hausdorff space and let $C^*(X)$ be the set of all bounded continuous functions on X. Then a sequence $\langle x_n \rangle$ converges in X to a point $x \in X$ iff for each $f \in C^*(X)$ we have $\lim f(x_n) = f(x)$. A sequence $\langle x_n \rangle$ is said to be <u>fundamental</u> whenever $\langle f(x_n) \rangle$ is a convergent sequence for all $f \in C^*(X)$. Clearly, a fundamental sequence $\langle x_n \rangle$ either converges in X or $\bigcup_{n \in \omega} \{x_n\}$ is a closed discrete subset of X. If each fundamental sequence converges in X, then X is said to be <u>sequentially complete</u>. Realcompact and normal spaces are sequentially complete ([3]).

<u>Proposition 1.</u> If $|X| = \omega$, then there is no convergent sequence in βX of the types IN-OUT or OUT-OUT.

PROOF. If $|X| = \omega$, then X is normal and hence sequentially complete. Thus no sequence $\langle x_n \rangle$ of points $x_n \in X$ can converge to a point $x \in \beta X - X$. Similarly, if $\langle x_n \rangle$ is a one-to-one sequence of points $x_n \in \beta X - X$, then $Y = X \cup \{x \in \beta X;\ x = x_n,\ n \in \omega\}$ is also a sequentially complete space. Thus $\langle x_n \rangle$ cannot converge in $\beta Y = \beta X$ to a point $x \in \beta Y - Y$. Consequently, the sequence $\langle x_n \rangle$ cannot converge in βX to a point $x \in \beta X - X$.

1. IN-OUT

Our construction of a space X in which there is a sequence $\langle x_n \rangle$ converging in βX to a point in $\beta X - X$ and in X no nontrivial sequence converges is based on the following idea.

First, let $\alpha > \omega$ be a cardinal number and let $Y = \omega \times (\alpha + 1)$. Define a topology for Y : all points $[n, \beta]$ for $n \in \omega$ and $\beta \in \alpha$ are isolated; a local base at $[n, \alpha]$ for $n \in \omega$ is formed by sets $\{[n,\alpha]\} \cup (K_n - S)$, where $K_n = \{[n,\beta] \in Y; \beta \in \alpha\}$ and S is a countable subset of K_n. Then Y is a completely regular Hausdorff space and for each $f \in C^*(Y)$ we have $f([n,\alpha]) = f([n,\beta])$ for all but countably many $\beta \in \alpha$. Note that no nontrivial sequence converges in Y.

Second, embed Y into a completely regular Hausdorff space X so that no nontrivial sequence converges in X, the sequence $\langle [n,\alpha] \rangle$ is a fundamental sequence in X, and the set $\{[n,\alpha] \in X; n \in \omega\}$ is a closed discrete subset of X. Then $\langle [n,\alpha] \rangle$ is an IN-OUT sequence.

At the Winter School we have presented the following space X, communicated to us by P. Simon.

Example 1. Consider the set $X = ((\omega+1) \times (2^c+1)) - \{[\omega, 2^c]\}$. Define a topology for X :

(i) All points $[n,\beta]$ for $n \in \omega$ and $\beta \in 2^c$ are isolated;

(ii) For $n \in \omega$ a local base at $[n, 2^c]$ is formed by sets $\{[n,\beta] \in X; \beta \in 2^c+1\} - S$, where S is a countable subset of the set $\{[n,\beta] \in X; \beta \in 2^c\}$;

(iii) Let h be a one-to-one mapping of 2^c onto $\{\mathcal{U} \in \mathcal{P}(\omega^*); |\mathcal{U}| = 2\}$ (for $\beta \in 2^c$, $h(\beta) = \{\mathcal{F}, \mathcal{G}\}$, where $\mathcal{F}$ and $\mathcal{G}$ are distinct uniform ultrafilters on ω). For $\beta \in 2^c$, $\{\mathcal{F}, \mathcal{G}\} = h(\beta)$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$, the sets $\{[\omega,\beta]\} \cup \{[n,\beta] \in X; n \in F \cup G\}$ form a local base at $[\omega, \beta]$.

It follows from the construction that X is a completely regular Hausdorff space in which no nontrivial sequence converges. Clearly, Y (with $\alpha = 2^c$) is a subspace of X. Further, $\langle [n, 2^c] \rangle$ is a fundamental sequence in X and $\{[n,2^c] \in X; n \in \omega\}$ is a closed discrete subset of X. Consequently, the sequence $\langle [n,2^c] \rangle$ converges in βX to a point in $\beta X - X$.

Here we present another construction of the space X (with no nontrivial convergent sequences) in which Y (with $\alpha = \kappa$) is embedded.

Example 2. In [1] it is shown that for

$\kappa = \min\{\delta;$ the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ is not $(\delta, \cdot, 2)$ distributive$\}$ there is a matrix $\{P_\alpha; \alpha \in \kappa\}$ such that the following conditions hold:

(1) P_α is a maximal almost disjoint family of subsets of ω;

(2) $\alpha < \beta$ implies P_β refines P_α;

(3) for each infinite subset x of ω there is $\alpha \in \kappa$ such that

$$|\{y \in P_\alpha;\ y \subseteq x\}| = c .$$

For each $\alpha \in \kappa$ define

$$\mathcal{F}_\alpha = \{x \subseteq \omega;\ |\{y \in P_\alpha; |y - x| = \aleph_0\}| < \aleph_0\} .$$

Clearly, $\mathcal{F}_\alpha$ is a filter on ω. Consider the set $X = ((\omega+1) \times (\kappa+1)) - \{[\omega, \kappa]\}$. The topology for X is defined analogously as in Example 1: (i) and (ii) remain and (iii) is replaced by

(iii)' for $\beta \in \kappa$, $F \in \mathcal{F}_\beta$ the sets $\{[\omega, \beta]\} \cup \{[n, \beta];\ n \in F\}$ form a local base at $[\omega, \beta]$.

Recall that $\omega_1 \leq \kappa \leq c < 2^c$, and so the cardinality of this space is $\kappa < 2^c$.

At the Winter School we have asked what is the minimal cardinality of the space X in which no nontrivial sequence converges and in X there is an IN-OUT sequence. In [4] it is shown that the minimal cardinality of such a space is ω_1. The construction is of the same type as in the above two examples. In the construction $\alpha = \omega_1$ and X is the set $((\omega+1) \times (\omega_1+1)) - \{[\omega, \omega_1]\}$ equipped with a topology in which neighborhoods of $[\omega, \beta]$, $\beta \in \omega_1$ are constructed via sums of Fréchet filters.

2. OUT-IN

Example 3. Consider the set $X = (\omega \times \omega) \cup \{\infty\}$ equipped with the following topology: all points $[n,m] \in \omega \times \omega$ are isolated; a local base at ∞ is formed by sets $\{\infty\} \cup (\{[m,n] \in \omega \times \omega; m > m_0, n > n_0\} - S)$, where $m_0, n_0 \in \omega$ and S is a subset of $\omega \times \omega$ containing finitely many points in each row and finitely many points in each column of $\omega \times \omega$. Then X is a countable completely regular Hausdorff space in which no nontrivial sequence converges. For each $n \in \omega$ $\beta\omega$ is homeomorphic to the closure in βX of the discrete closed set $K_n = \{n\} \times \omega$, the homeomorphism being fixed on ω. It is easy to see that if $x_n \in \mathrm{cl}_{\beta X} K_n - K_n$, then the sequence $\langle x_n \rangle$ converges in βX to the point ∞. Since X is countable, it follows from Proposition 1 that there are no (nontrivial) IN-OUT or OUT-OUT sequences in βX.

3. OUT-OUT

In our talk at the Winter School we have presented a space (having no nontrivial convergent sequences) for which there are both IN-OUT and OUT-OUT sequences. The space itself has been constructed by tying together a sequence of distinct copies of the space X from Example 1. We have also announced that we are able to construct a space (having cardinality c) for which there are only OUT-OUT sequences. We present the construction below (Example 4). After the Winter School, during a short visit of W. S. Watson in Košice, we have constructed several spaces (with no nontrivial convergent sequences) having cardinality ω_1 for which there are only OUT-OUT sequences. This, together with Proposition 1, shows that ω_1 is the minimal cardinality of such spaces. For details see [4].

<u>Example 4</u>. In this construction we use the following observation about ω^*. It is known ([2]) that each point of ω^* is a c-point (e.g. ekvivalently, for each nontrivial ultrafilter $j = \{x_\alpha ; \alpha \in c\}$ on ω there is an almost disjoint refinement (i.e. a system $\{y_\alpha ; \alpha \in c\}$ such that $y_\alpha \subseteq x_\alpha$ and $\alpha \neq \beta$ implies $|y_\alpha \cap y_\beta| < \aleph_0$)). A nontrivial ultrafilter j on ω is said to be a <u>σ-c-point</u> if the following holds: Let $\{X_\alpha ; \alpha \in c\} = [j]^\omega$ be an enumeration of all countable subsets of j. Then there is an almost disjoint family $\{y_\alpha ; \alpha \in c\}$ on ω such that for each $\alpha \in c$ and each $x \in X_\alpha$ we have $y_\alpha \subseteq^* x$ (modulo finite). Using a slight modification of Hindman's proof (see [5]) of the existence of c-points we can prove the existence of a σ-c-point.

<u>Proposition 2</u>. There are always σ-c-points in ω^*; assuming CH or MA or RP (Roitman principle), all points of ω^* are σ-c-points.

We do not know whether in ZFC each point of ω^* is a σ-c-point.

Construction. Let j be a σ-c-point and let X_α, y_α be as above. For $\alpha \in c$, enumerate $X_\alpha = \{x_n^\alpha ; n \in \omega\}$ and take the product $R_\alpha = \prod_{n\in\omega} (x_n^\alpha \cap y_\alpha)$. Then R_α is isomorphic to ${}^\omega\omega$. As κ (from Example 2) is less or equal to the smallest size of an unbounded family in ${}^\omega\omega$, ordered modulo finite (see [1]), there is a strictly increasing sequence of one-to-one functions $\{f_\beta^\alpha ; \beta < \kappa\} \subseteq R_\alpha$. Clearly, for $[\alpha, \beta] \neq [\gamma, \delta]$ we have $|f_\beta^\alpha \cap f_\delta^\gamma| < \aleph_0$.

Consider the set $X = \omega \times \omega \cup c$. Define a topology for X:

(i) All points $[n,m]$ for $n, m \in \omega$ are isolated;

(ii) Let h be a one-to-one mapping from c onto $c \times \kappa$ and let α, β, γ be such that $h(\gamma) = [\alpha, \beta]$. For $F \in \mathcal{F}_\beta$ (the very filter from Example 2) the sets $\{\gamma\} \cup \{[n, f_\beta^\alpha(n)];\ n \in F\}$ form a local base at the point γ.

Then the closure of the set $V_n = \{[n,m];\ m \in \omega\}$ in βX contains j_n, the copy of the σ-c-point j. Then $\langle j_n \rangle$ is a fundamental sequence and βX is a "pure OUT-OUT" space.

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