BOOLEAN GAMES - CLASSIFYING STRATEGIES AND OMITTING CARDINALITY ASSUMPTIONS

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ABSTRACT. We deal with a transfinite game on Boolean algebras introduced by T. Jech. The game yields a fine method for handling κ-closed dense subsets of Boolean algebras. We prove (without set-theoretical assumptions) the existence of a γ^+-closed dense subset for a certain type of Boolean algebras determined in the game of an uncountable length γ - a generalization of some results by M. Foreman. We investigate relationship between certain cardinal characteristics of Boolean algebras, discuss the existence of positional strategies of trees, and give a couple of problems concerning the partialy ordered set of all strategies.

1. Introduction and notation. In terminology we generally follow [8],[9],[11], but some notions are introduced in this section. Let B be an atomless Boolean algebra and α an ordinal number. Consider the following transfinite game $\mathcal{G}^{I}(B,\alpha)$, introduced by T.Jech in [5], between two players White and Black. Let White and Black define a decreasing sequence

(1) $$w_0 \geq b_0 \geq w_1 \geq \dots \geq w_\xi \geq b_\xi \geq \dots$$

of nonzero elements of B of length $\leq \alpha$ by taking turns defining its entries. I.e., first White chooses a nonzero $w_0 \in B$. Then Black chooses a nonzero $b_0 \leq w_0$. Then White chooses nonzero $w_1 \leq b_0$ ... The play is won by Black if the sequence (1) has nonzero lower bound and length α; else the White wins.

The game $\mathcal{G}^{II}(B,\alpha)$ (see [4],[3]) is defined in exactly the same way as the game $\mathcal{G}^{I}(B,\alpha)$, except that the player Black moves first at limit stages, i.e. the play of $\mathcal{G}^{II}(B,\alpha)$ looks like

$$w_0, b_0, w_1, b_1, \dots, b_\omega, w_\omega, b_{\omega+1}, w_{\omega+1}, \dots, b_\xi, w_\xi, \dots$$

T. Jech in [5] proved that if the algebra B has a κ^+-closed dense subset, then the player Black has a winning strategy in the game $\mathcal{G}^{I}(B,\kappa)$, $\mathcal{G}^{II}(B,\kappa)$. He also formulated the problem whether the inverse implication holds, i.e., does the existence of a winning strategy for the Black in the game $\mathcal{G}^{I}(B,\kappa)$ ($\mathcal{G}^{II}(B,\kappa)$) imply that the algebra B has a κ^{+}-closed dense subset? The problem for $\kappa = \omega$ was investigated in [5], [3], [8] and [11]. For $\kappa = \omega_1$, C. Gray in [4] has constructed an algebra E such that Black wins $\mathcal{G}^{II}(E, \omega_1)$ and E has no ω_2-closed dense subset (nothing similar for the game $\mathcal{G}^{I}$ is known). M. Foreman in [3] proved that if $d(B) = \lambda^{+} = ND(B)$, where $ND(B)$ denotes the nondistributivity of B, Black wins $\mathcal{G}^{I}(B,\gamma)$ and $\lambda^{<\gamma} = \lambda$, then the algebra B has a γ^{+}-closed dense subset. We show that the saturatedness of such an algebra can be either λ^{+} or λ^{++} and in the first case the same conclusion holds without the assumption about the cardinal - exponentation (Theorem 1).

We say that $D \subseteq B^{+}$ is a $\underline{\lambda\text{-closed dense subset}}$ of algebra B (we say sometimes base instead of dense subset) if $(\forall x \in B^{+})$ $(\exists y \in D)(y \leq x)$ and for every decreasing sequence $\{a_\alpha : \alpha < \tau\} \subseteq D$ of the length $\tau < \lambda$ there is a $y \in D$ such that $y \leq a_\alpha$ for each $\alpha < \tau$. Define:

$d(B) = \min \{|D| : D$ is a dense subset of $B\}$,

$ND(B) = \min \{\delta : B$ is not $(\delta, \cdot, 2)$-distributive$\}$,

$\nabla hsat(B) = \min\{\kappa : (\forall x \in B^{+})($there is no partition of B_x of size $\kappa)\}$,

$\Delta hsat(B) = \sup\{\kappa : (\forall x \in B^{+})($there is a partition of B_x of size $\kappa)\}$,

$\nabla cds(B) = \min\{\kappa :$ there is no κ-closed dense subset of $B\}$,

$\Delta cds(B) = \sup\{\kappa :$ there is a κ-closed dense subset of $B\}$,

$\nu_1(B) = \sup(\mathcal{H}^{I}(B)) = \sup\{\alpha :$ Black wins $\mathcal{G}^{I}(B, \alpha)\}$,

$\eta_1(B) = \min(\mathfrak{Z}^{I}(B)) = \min\{\alpha :$ White wins $\mathcal{G}^{I}(B, \alpha)\}$,

analogously we define ν_2, η_2, $\mathcal{H}^{II}$, $\mathfrak{Z}^{II}$ for the game $\mathcal{G}^{II}$.

It is known that $\eta_2(B) = ND(B)$ (see [3]) and that ν_1, ν_2, η_1 are regular cardinal numbers (see [11]).

2. Omitting cardinality assumptions in the game of uncountable length.

The following facts may be belong to folklore.

Proposition. For every atomless Boolean algebra B the following hold:

(i) $\Delta hsat(B) \leq d(B)$ and $\nabla hsat(B) \leq d(B)$ does not hold;

(ii) $ND(B) \leq \nabla hsat(B)$ and $ND(B) \leq \Delta hsat(B)$ does not hold;

(iii) $\Delta cds(B) \leq ND(B)$ and $\nabla cds(B) \leq ND(B)$ does not hold;

(iv) $\Delta cds(B) \leq \nu_1(B)$ and $\nabla cds(B) \leq \nu_1(B)$ does not hold;

(v) $ND(B) \leq d(B)$;

(vi) $\nabla hsat(B) \leq (\Delta hsat(B))^{+}$ and $\nabla cds(B) \leq (\Delta cds)^{+}$.

PROOF. The negative assertions in (i) - (iv) are trivial.

(i) Follows easily, for if P is a partition of B then $|P| \leq d(B)$.

(ii) Let $\delta = \nabla hsat(B) < ND(B)$. As B is atomless, there is a matrix $\mathcal{M} = \{P_\alpha : \alpha < ND(B)\}$ consisting of maximal partitions of B such that $\alpha < \beta$ implies P_β strictly refines P_α . Then for each $x \in P_\delta$ the set $\{y_\alpha \in P_\alpha : \alpha < \delta \ \& \ x \leq y_\alpha\}$ is a strictly decreasing tower of algebra B and $\{y_{\alpha+1} - y_\alpha : \alpha < \delta\}$ is a partition of B of size δ. Contradiction.

(iii) If B has a κ-closed dense subset, then $\kappa \leq ND(B)$. (Follows also from (iv) and $\nu_1(B) \leq \eta_2(B) = ND(B)$, see [11]).

(iv) See [5].

(v) Assume $D = \{x_\alpha : \alpha < \delta\}$ is a base of B and $\delta < ND(B)$. Let P be a strict refinement of the matrix $\{\{x_\alpha; -x_\alpha\} : \alpha < \delta\}$. For $x \in P$, take $x_\alpha \in D$ with $x_\alpha \leq x$. Contradiction.

(vi) Obvious.

The following Lemma shows that the existence of certain algebras has influence on the exponentation of cardinal numbers.

Lemma. Assume that B is a Boolean algebra such that

$$\kappa < \nabla hsat(B) \quad \text{and} \quad \gamma \in \mathcal{H}^{II}(B) .$$

Then $\kappa^\gamma \leq \Delta hsat(B)$ and $\kappa^\gamma < \nabla hsat(B)$.

The Proof is analogous as that of Corollary 1 in [11].

The next theorem generalizes some results of M. Foreman ([3]).

Theorem 1. Assume that B is anatomless Boolean algebra such that $d(B) = ND(B) = \lambda^+$ and Black wins $\mathcal{G}^I(B,\gamma)$. Then:

(1) $\Delta hsat(B) < \nabla hsat(B)$.

(2) Either $\Delta hsat(B) = \lambda^+$ and $\nabla hsat(B) = \lambda^{++}$, or $\Delta hsat(B) = \lambda$ and $\nabla hsat(B) = \lambda^+$.

(3) If $\Delta hsat(B) = \lambda$, then the algebra B has a γ^+-closed dense subset.

PROOF. (1) If $\Delta hsat(B) = \nabla hsat(B)$, then from (i) and (ii) in Proposition we have $\Delta hsat(B) = \nabla hsat(B) = \lambda^+$. But in this case $\nabla hsat(B)$ should be a weakly inaccessible cardinal number (see [7]). Contradiction.

(2) As $\nabla hsat(B) \leq (\Delta hsat(B))^+$, (2) follows from (i) and (ii) in Proposition.

(3) Aplying Lemma, $\lambda < \nabla hsat(B)$ and $\gamma \in \mathcal{H}^I(B)$ imply $\lambda^\gamma = \lambda$. Then $\lambda \leq \lambda^{<\gamma} \leq \lambda^\gamma = \lambda$ shows that the additional Foreman's set-theoretical assumption is for algebras in question granted.

Remarks. To prove a similar result for algebras having bigger density we may be tempted to use the more general construction of base matrices from Lemma 2 of [11]. But if algebra B is $(\lambda^+, \cdot, \kappa)$-nowhere distributive, $\gamma \in \mathcal{W}^I$, $d(B) = \kappa^\gamma$ and $\Delta hsat(B) = \lambda$, we obtain only $d(B) \leq \lambda^+$!

It might be in place to call the reader's attention to an interesting "inverse" exponentation of cardinals in the Theorem 6 of [9].

Note that for $\Delta hsat(B) = \lambda^+$ Theorem 1 implies $(\lambda^+)^\gamma = \lambda^+$. Then take $\rho = \min \{\min \{\tau \leq \gamma : \lambda^\tau = \lambda^+\}, \gamma\}$. If $\rho > \omega_0$, then the algebra B has a ρ^+-closed dense subset.

The case $\Delta hsat(B) = \lambda^+$ will be further discussed in § 3 using positional strategies.

3. Classifying strategies and problems. The importance of classifying different types of strategies was shown in [11], namely the Gray's trick for constructing determined algebras without closed dense subset does not work below ω_1 .

Definition ([5]). We say that Black has a <u>positional winning strategy</u> in the game $\mathcal{G}^I(B,\kappa)$ if there is a function $\rho : B^+ \to B^+$ such that Black wins every play of length κ in which he follows ρ : $w_0, \rho(w_0), w_1, \rho(w_1), \dots, w_\omega, \rho(w_\omega), \dots, w_\xi, \rho(w_\xi), \dots$; $\xi < \kappa$. For the motivation of the following definition see [8],[9] and [11]. Moreover, we mention the following point of view. There is a lot of games which finish after reaching the winning position (e.g. chess), or at a certain point an evaluation is made to decide the game (e.g., Mycielski's game, some topological games). Jech's game has one interesting feature: the Black's victory in fact says that we can continue the play. This enables us to study a specific type of questions that are not possible for other games:

- the questions about sets $\mathcal{W}, \mathcal{Z}$ of ordinals for which Black (White) has a winning strategy (see [11]),

- the questions about relations between strategies for games of different length (e.g. does a strategy σ for the game $\mathcal{G}(B,\alpha)$ with $\alpha > \beta$ prolongate the strategy ρ for the game $\mathcal{G}(B,\beta)$?).

So our Boolean game gives us motivation for studying such aspects for other games. For instance, we can ask (perhaps an obscure question): How long, in chess, can Black or White continue the play ?

Definition ([8]). We say that the player Black has <u>a simultaneous winning strategy</u> in the game $\mathcal{G}^I$ ($\mathcal{G}^{II}$, respectively) on algebra B if there is one strategy

$$\sigma : \bigcup \{{}^\beta B : \beta < \nu_1(B)\} \longrightarrow B^+$$

such that σ is winning for Black in each game $\mathcal{G}^I(B,\alpha)$ for

$\alpha < \nu_1(B)$ ($\mathcal{G}^{II}(B, \alpha)$ for $\alpha < \nu_2(B)$, respectively).

Consider the set

$$\wp^{I}(B) = \{\rho ; \rho \text{ is a winning strategy for Black in } \mathcal{G}^{I}\}$$

and a partial ordering of $\wp^{I}(B)$:

$$\rho \le \tau \text{ if } \rho \supseteq \tau$$

Then $(\wp^{I}(B), \le)$ is a tree of length $\nu_1(B)$ (analogously for $\mathcal{G}^{II}$). Observe that Black has a simultaneous strategy in $\mathcal{G}^{I}(B)$ if and only if in the tree $(\wp^{I}(B), \le)$ there is a branch of the length $\nu_1(B)$.

Games played on a partialy ordered set P and on the Boolean completion RO(P) are equivalent (see [5]). We shall consider the special case when P is a tree. It concerns algebras which have a base matrix - i.e. a base which forms a tree in the natural ordering of the algebra B .

Theorem 2. Assume that T is a tree of height κ, cf$(\kappa) > \omega$ and the player Black has a positional winning strategy in the game $\mathcal{G}^{I}(T, \gamma)$ with $\gamma > \omega_0$. Then T has a γ^{+}-closed dense subset.

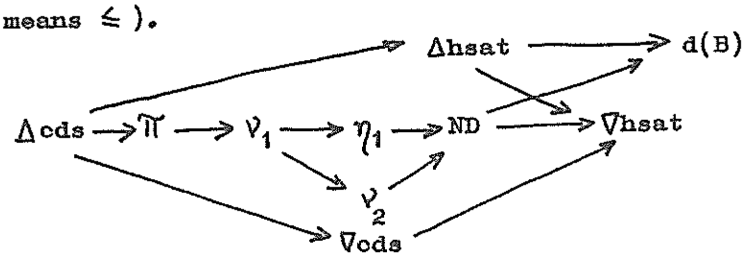
PROOF. Following the Foreman's proof (see [3]), for each $t \in T$ we will define a $t^{*} \in T$, $t^{*} \le t$ with the property, that if $\bar{s}$ is a partial play towards t^{*} and $t' \in T$ with $\inf \bar{s} \geq t' > t^{*}$, then there is a partial play towards t^{*} extending $\bar{s}, \bar{s}'$ such that $t' \geq \inf \bar{s}' \geq t^{*}$. Using a positional strategy π for $\mathcal{G}^{I}(T, \gamma)$ define $t_0 = t$ and for $n \in \omega$ $t_{n+1} = \pi(t_n)$. The sequence $\{t_n ; n \in \omega\}$ has a nonzero lower bound - take one with minimal rank in the tree T and denote it by t^{*} . Now the proof proceeds as in [3].

We remark, that Theorem 2 deals with a larger class of algebras than that treated in Theorem 1.

The following problem seems to be important.

Problem. Does the existence of a winning strategy for Black on a tree T imply the existence of a positional winning strategy for Black on T ?

Consider the following extensions of the representation problem from [11]. The results of our Proposition, [11] and further folklore results are shown below on an oriented graph (arrow $\longrightarrow$ means $\le$).

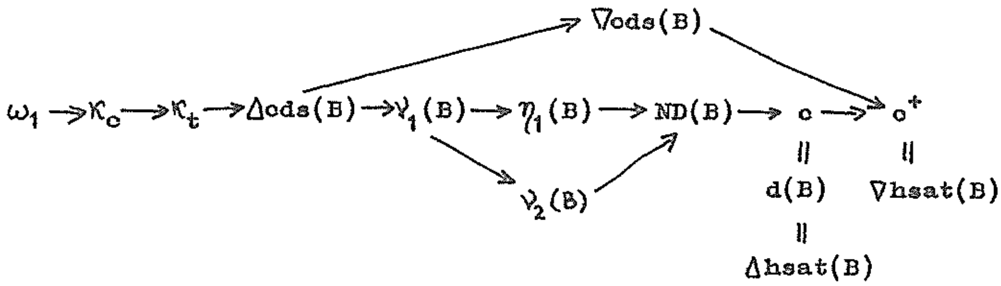


Question. If we prescribe to each vertex of our graph a cardinal number such that inequalities are fulfilled and $\pi, \nu_1, \eta_1, \nu_2$, ND, ∇hsat are regular, $\nabla\text{cds} \leq (\Delta\,\text{cds})^+$, $\nabla\,\text{hsat} \leq (\Delta\,\text{hsat})^+$, does then there exist a Boolean algebra B such that all its characteristics are as prescribed (here $\pi = \sup\{\alpha :$ Black has a positional winning strategy in $\mathcal{G}^{\mathrm{I}}(B, \alpha)\}$) ? Moreover, we can ask whether such an algebra exists if we prescribe the existence (or nonexistence) of the simultaneous winning strategy for $\mathcal{G}^{\mathrm{I}}$ of the length $\nu_1(B)$ and for $\mathcal{G}^{\mathrm{II}}$ of the length $\nu_2(B)$.

The special case of this representation problem arises if $B = \mathcal{P}(\omega)/\text{fin}$ - the algebra of power set of the set of all natural numbers modulo the ideal of finite sets. We define (see also [1])

$$\kappa_c = \min\{|F| : F \subseteq \mathcal{P}(\omega)/\text{fin} \text{ is centered } \& \bigwedge F = \mathbb{O}\},$$
$$\kappa_t = \min\{|T| : T \subseteq \mathcal{P}(\omega)/\text{fin} \text{ is a tower and } \bigwedge T = \mathbb{O}\} .$$

In this case the graph looks like ($B = \mathcal{P}(\omega)/\text{fin}$):



In [1] it is showed that ND(B) can be strictly smaller than c . In [2] Con (ZFC + $\kappa_c <$ ND) is proved and in [7] it is proved that $\kappa_c = \omega_1$ implies $\kappa_t = \omega_1$ and κ_c is a regular cardinal number. This together with Dordal's metatheorem ([2]) gives Con (ZFC+ $\kappa_t <$ ND). Is it consistent that some other inequalities are strict ? In particular, is it consistent that

$$\kappa_t < \Delta\text{cds}\ (\mathcal{P}(\omega)/\text{fin})\ \ ?$$

At the end we mention the following problem, presented at the Logic Colloquium '82 ([10]).

Let B be a Boolean algebra. Put

ΔIP(B) = sup $\{\kappa$: there is a κ^+-closed dense subset of B$\}$,

∇IP(B) = min $\{\kappa$: there is no κ^+-closed dense subset of B$\}$.

The following function describes the global behaviour of our game: for a cardinal number λ define

$b^{\mathrm{I}}(\lambda) = \min\{\Delta\text{IP}(B) :$ B is such that $\lambda \in \mathcal{H}^{\mathrm{I}}(B)\}$

(analogously b^{II})

Problem. 1. Does $(\forall\kappa)(\exists\lambda)(b^{\circ}(\lambda) \geq \kappa)$ hold ?

2. Is there a regular cardinal number ϑ such that for each $\kappa < \vartheta$ there is a $\lambda < \vartheta$ such that $b(\lambda) \geq \kappa$?

Note that $b(\lambda) \leq \lambda$ and the failure of the implication "the existence of a strategy for Black implies the existence of a closed dense subset" causes that the function b is regressive. This makes the questions more interesting.

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