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GAME PROPERTIES OF BOOLEAN ALGEBRAS

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Abstract: We deal with the transfinite game on Boolean algebras introduced by T.Jech at the Logic Colloquium '77. We prove that if the density of a Boolean algebra B is 2^ω and 2^ω is smaller than the first weakly inaccessible cardinal number, and if the player Black has a winning strategy, then B has a σ-closed dense subset (this generalizes some results obtained by M.Foreman). Moreover, we investigate properties of the sets of ordinal numbers of the lengths of games for which Black, respectively White, has a winning strategy. The results suggest that the game is "cardinal-type" rather then "ordinal-type".

Key words: Games on Boolean algebras, σ-closed dense subset, cardinal characteristics of Boolean algebras.

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§ 1. Introduction. Let B be a Boolean algebra and α an ordinal number. Consider the following transfinite game $\mathcal{G}^I(B,\alpha)$, introduced by T.Jech in [6], between two players White and Black. Let White and Black define a decreasing sequence

(1) $\qquad w_0 \geq b_0 \geq w_1 \geq b_1 \geq \dots \geq w_\xi \geq b_\xi \geq \dots$

of nonzero elements of B of length $\leq \alpha$ by taking turns defining its entries i.e. first White chooses a nonzero $w_0 \in B$. Then Black chooses a nonzero $b_0 \leq w_0$. Then White chooses nonzero $w_1 \leq b_0 \dots$ The play is won by Black if the sequence (1) has nonzero lower bound and length α; else the White wins.

The game $\mathcal{G}^{II}(B,\alpha)$ (see [5],[4]) is defined in exactly the same way as the game $\mathcal{G}^{I}(B,\alpha)$ except that the player Black moves first at limit stages, i.e. the play of $\mathcal{G}^{II}(B,\alpha)$ looks like

$$w_0, b_0, w_1, b_1, \dots, b_\omega, w_\omega, b_{\omega+1}, w_{\omega+1}, \dots, b_\xi, w_\xi, \dots$$

T. Jech in [6] proved that if the algebra B has a σ-closed dense subset, then the player Black has a winning strategy in the game $\mathcal{G}(B,\omega) = \mathcal{G}^{I}(B,\omega) = \mathcal{G}^{II}(B,\omega)$ and formulated the problem whether the inverse implication holds i.e., does the existence of a winning strategy for the Black in the game $\mathcal{G}(B,\omega)$ imply that the algebra B has a σ-closed dense subset? The problem was investigated in [4], [5], [10] and [11]. M. Foreman ([4]) proved that if the density of a Boolean algebra B is ω_1 and Black wins the game $\mathcal{G}(B,\omega)$ then algebra B has a σ-closed dense subset. We show in § 2 that the assumptions of this Foreman's theorem implies CH and we prove the same conclusion under the assumption that the density of B is 2^{ω}. C. Gray in [5] (cited in [4]) constructed a Boolean algebra E such that Black wins $\mathcal{G}^{II}(E,\omega_1)$ but does not win $\mathcal{G}^{I}(E,\omega_1)$ and hence E has no ω_2-closed dense subset. We show that such a phenomenon cannot occur below ω_1. To be more precise, for a Boolean algebra B denote

$$\mathcal{B}^{I}(B) = \{\alpha : \text{Black wins } \mathcal{G}^{I}(B,\alpha)\}$$
$$\mathcal{Z}^{I}(B) = \{\alpha : \text{White wins } \mathcal{G}^{I}(B,\alpha)\}$$
$$\mathcal{B}^{II}(B) = \{\alpha : \text{Black wins } \mathcal{G}^{II}(B,\alpha)\}$$
$$\mathcal{Z}^{II}(B) = \{\alpha : \text{White wins } \mathcal{G}^{II}(B,\alpha)\}.$$

We prove that sup $\mathcal{B}^{I}$ and sup $\mathcal{B}^{II}$ are regular cardinal numbers and hence $(\mathcal{B}^{II} - \mathcal{B}^{I}) \cap \omega_1 = \emptyset$. In [4] it is proved that min $\mathcal{Z}^{II}(B)$ is the minimal cardinal number δ such that the algebra B is not $(\delta, \cdot, 2)$-distributive. We prove that min $\mathfrak{Z}^{I}(B)$ is a regular cardinal number. These results show that, as far we are interested in the existence of the winning strategies for Black respectively White, it suffices to consider games the length of which is a cardinal number.

Furthermore, it follows that to break down the inverse implication mentioned above we have to look for finer methods, e.g., the one introduced in [10], based on the notion of a simultaneous strategy.

Preliminaries. In this paper Boolean algebras are assumed to be complete, without atoms. Let B be a complete Boolean algebra, $\mathbb{O}$ a zero of B. Then $B^{+} = B - \{\mathbb{O}\}$, for $x \in B^{+}$ B_x is the partial algebra. We say that $D \subseteq B^{+}$ is a λ-closed dense subset of algebra B (we say sometimes base instead of dense subset) if $(\forall x \in B^{+})(\exists y \in D)(y \leq x)$ and for every decreasing sequence $\{a_\alpha : \alpha < \tau\} \subseteq D$ of the length $\tau < \lambda$ there is an $y \in D$ such that $y \leq a_\alpha$ for each $\alpha < \tau$ (σ-closed means ω_1-closed). d(B) denotes the density of B, i.e.

min $\{|X| ; X \subseteq B^{+}$ is a dense subset of $B\}$. hsat(B) denotes the hereditary saturatedness, where hsat(B) $= \kappa$ iff

$(\forall x \in B^{+})(\forall \lambda < \kappa)$(there is a partition P of B_x with $|P| \geqslant \lambda$). By P,Q,W we usualy denote a maximal partition of B. A system $\Theta = \{P_\alpha ; \alpha < \kappa\}$ is called a matrix, P_α's are columns of Θ and $x \in P_\alpha$ is an element of the matrix Θ. For $x \in B^{+}$ put

$x \wedge\wedge P = \{y \wedge x;\ y \in P\} \cap B^{+}$. $P \ll Q$ denotes that P refines Q, i.e. $(\forall x \in P)(|x \wedge\wedge Q| \leq 1)$. A matrix Θ is said to be monotone provided $\alpha < \beta$ implies $P_\beta \ll P_\alpha$. Note that if Θ is monotone, then $(\bigcup \Theta, \leq)$ forms a tree. The algebra B is said to be

$(\kappa,\cdot,\lambda)$-distributive (see [12],[6]) provided for any matrix $\Theta = \{P_\alpha ; \alpha < \kappa\}$ there is a maximal partition P of B such that $(\forall x \in P)(\forall \alpha < \kappa)(|x \wedge\wedge P_\alpha| < \lambda)$ and B is called $(<\kappa,\cdot,\lambda)$-distributive if for every $\delta < \kappa$ B is $(\delta,\cdot,\lambda)$-distributive. Further B is said to be nowhere $(\kappa,\cdot,\lambda)$-distributive if for each $x \in B^+$ the algebra B_x is not $(\kappa,\cdot,\lambda)$-distributive. Recall that B is nowhere $(\kappa,\cdot,\lambda)$-distributive iff there is a matrix $\Theta = \{P_\alpha ; \alpha < \kappa\}$ such that for each $x \in B^+$ there is some $\alpha < \kappa$ with $|x \wedge\wedge P_\alpha| \geq \lambda$. In this case we say that Θ is a matrix witnessing to the nowhere $(\kappa,\cdot,\lambda)$-distributivity of the algebra B_x.

Recall that $(\kappa,\cdot,2)$-distributivity of B is equivalent to the following condition on the algebra B: For each matrix $\{P_\alpha ; \alpha < \kappa\}$ of B we have

$$\bigwedge_{\alpha<\kappa} \bigvee P_\alpha = \bigvee \{\bigwedge_{\alpha<\kappa} f(\alpha) : f \in \prod_{\alpha<\kappa} P_\alpha\}$$

Symbol $\mathcal{H}^{\cdot}$ stands for any of $\mathcal{H}^{I}$ and $\mathcal{H}^{II}$; similarly for $\mathcal{Z}^{\cdot}$ and $\mathcal{G}^{\cdot}$. The winning strategy for the player Black in the game $\mathcal{G}^{I}(B,\alpha)$ is a function $f : \bigcup_{\beta<\alpha} {}^{\beta}B \to B$ such that Black wins every play in which he follows f, i.e., each play

$$w_0, f(w_0), w_1, f(w_0,w_1), \ldots, w_\beta, f(w_0,w_1,\ldots,w_\beta), \ldots$$

is won by Black i.e. $\bigwedge_{\beta<\alpha} w_\beta \neq \mathbb{O}$. We admit that f can be defined possibly on a greater domain, e.g. if $\alpha < \beta$ and σ is a winning strategy for Black in $\mathcal{G}^{\cdot}(B,\beta)$ then σ is also a winning strategy for Black in $\mathcal{G}^{\cdot}(B,\alpha)$. Sometimes it is more convenient to write the infinite sequence $\{b_\xi : \xi < \alpha\}$ more explicitly, e.g.,

$$\{b_0, b_1, \ldots, b_n, b_{n+1}, \ldots, b_\omega, b_{\omega+1}, \ldots, b_\xi, \ldots : \xi < \alpha\}$$

and an element of this sequence with double indexes $\delta_{\xi+1^{+1}}$

we compromisely write as $b(\delta_{\xi+1}+1)$ instead of $b_{\delta_{\xi+1}+1}$.

The course of a play $w_0 \geqslant b_0 \geqslant w_1 \geqslant b_1 \geqslant \dots \geqslant w_\alpha \geqslant b_\alpha \geqslant \dots$ we often ilustrate by a picture like this

White	Black
w_0	
	b_0
w_1	
	b_1
$\vdots$	
w_α	
	b_α
$\vdots$	

For unexplained notation and terminology we refer to [7].

§ 2. Partial affirmative results

Corollary 1. Assume that B is a complete Boolean algebra without atoms and Black has a winning strategy in the game $\mathcal{G}(B,\omega)$. Then $\mathrm{hsat}(B) > 2^\omega$.

Proof. Let us pick an arbitrary $x \in B^+$ and σ a winning strategy for the Black and construct a binary tree

$$T = \{x_f : f \in \bigcup_{n<\omega} {}^n 2\} \subseteq B_x^+ \text{ such that}$$

(i) $f \subsetneq g \to x_g \leq \sigma(x_{f\restriction 0}, x_{f\restriction 1}, \dots, x_f)$, and

(ii) $x_{f^\frown 0} \wedge x_{f^\frown 1} = \mathbb{O}$.

Easily for any $f \in {}^\omega 2$ the intersection $y_f = \bigwedge_{n<\omega} x_{f\restriction n}$ is distinct from $\mathbb{O}$, and for $f \neq g$ we have $y_f \wedge y_g = \mathbb{O}$ and $y_f \leq x$. Hence $\mathrm{sat}(B_x) > 2^\omega$.

Remark 1. As already mentioned in the introduction, M. Foreman ([4]) proved that if Black wins $\mathcal{G}(B,\omega)$ and $d(B) = \omega_1$, then B has a σ-closed dense subset. From our Corollary we get $2^{\omega} < \mathrm{hsat}(B) \leq d(B)^{+} = \omega_2$ and therefore $2^{\omega} = \omega_1$ must hold. A natural question arises: what happens if $d(B) = 2^{\omega} > \omega_1$. In what follows we provide the answer.

Remark 2. Every Souslin algebra B is not determined. This follows from the fact that B is $(\aleph_0,.,2)$-distributive (i.e. White does not win $\mathcal{G}(B,\omega)$ (see [6])) and $\mathrm{hsat}(B) = \omega_1$ (i.e. according to our Corollary, Black does not win $\mathcal{G}(B,\omega)$).

Definition. A matrix $\textcircled{R} = \{P_\alpha \ ; \alpha < \kappa\}$ of algebra B is called a base matrix of B of length κ if $\bigcup \textcircled{R} = \bigcup\{P_\alpha \ ; \alpha < \kappa\}$ is a base of the algebra B (i.e. a dense subset of B).

The notion of a base matrix was introduced in [1] and for "strategic" constructions used in [10], [11] and [4].

Lemma 2. Assume B is nowhere $(\kappa,.,\lambda)$-distributive and $(<\kappa,.,2)$-distributive Boolean algebra such that $d(B) = \lambda^{\aleph_0}$ and Black has a winning strategy in the game $\mathcal{G}(B,\omega)$. Then B has a monotone base matrix of the length κ and $\kappa \geq \omega_1$.

Proof. The proof can be obtained by a slight modification of the proofs of Lemma 6 and Theorem 7 in [11]. Hint: Take a monotone matrix $\textcircled{H}$ witnessing to the nowhere $(\kappa,.,\lambda)$-distributivity. Using the strategy for Black one can construct a λ-ary tree of the length $\aleph_0$ analogously as in Corollary 1. Now, each nonzero element of the dense subset of B of size $\lambda^{\aleph_0}$ intersects $\lambda^{\aleph_0}$ different elements of the matrix. Take a one-to-one embedding of the dense subset of B into the elements of the matrix $\textcircled{H}$ and "close" the matrix under intersections and complements. This way we get the desired base matrix.

Foreman's construction of a σ-closed dense subset has two steps. First he constructs a base matrix of nonlimit length. Second, from this base matrix he constructs a σ-closed dense subset. Generalizing the first step and using the second Foreman's step unchanged, we get the following.

Theorem 3. Assume that algebra B is $(\kappa,.,2)$-distributive and nowhere $(\kappa^+,.,\lambda)$-distributive, $d(B) = \lambda^{\aleph_0}$ and Black wins $\mathcal{G}(B,\omega)$. Then B has a σ-closed dense subset.

To reach the continuum we use the technique of the partitioning algebra into factors homogeneous in some cardinal characteristics (cf. Pierce [8], Bukovský [2]).

Lemma 4. For $x \in B^+$ put $\pi(x) = \kappa$ whenever the algebra B_x is $(<\kappa,.,2)$-distributive and nowhere $(\kappa,.,2)$-distributive (undefined otherwise). Then

(a) $\mathcal{D} = \{x: \pi(x)$ is defined$\}$ is a dense subset of B,

(b) $(\forall x \in \mathcal{D})(\pi(x)$ is a regular cardinal number smaller or equal to the density of $B)$.

Proof. (a) Pick some $x \in B^+$. Put

$$\kappa = \min\{\delta: B_x \text{ is not } (\delta,.,2)\text{-distributive}\}.$$

Then there is a matrix $\{P_\alpha : \alpha < \kappa\}$ such that $\bigwedge_{\alpha<\kappa} \bigvee P_\alpha = x$ and $z = \bigvee\{\bigwedge_{\alpha<\kappa} f(\alpha): f \in \prod_{\alpha<\kappa} P_\alpha\} \lneq x$.

The algebra B_{x-z} is $(<\kappa,.,2)$-distributive and nowhere $(\kappa,.,2)$-distributive, i.e. $x-z \leq x$ and $\pi(x-z) = \kappa$.

(b) The proof that $\pi(x)$ is regular is straightforward. Assume that $d(B_x) \leq d(B) < \pi(x)$ and $\{u_\alpha : \alpha < d(B_x)\}$ is a base of B_x. Put $P_\alpha = \{u_\alpha, -u_\alpha\}$. The size of matrix $\{P_\alpha : \alpha < d(B_x)\}$ is smaller than $\pi(x)$ and, by the distributivity of B_x, there is a common refinement P. Take $x \in P$ and a $y \in B^+$, $y \lneq x$ (B is atomless). But, there is an element of the base u_α such that $u_\alpha \leq y \lneq x \leq u_\alpha$. Contradiction.

Theorem 5. Assume that B is a Boolean algebra such that Black wins the game $\mathcal{G}(B, \omega)$ and $d(B) = 2^\omega$ and $2^\omega <$ 1st weakly inaccessible cardinal number. Then B has a σ-closed dense subset.

Proof. Take $\mathcal{D} = \mathrm{dom}(\pi)$, the dense set from Lemma 4, and take a maximal disjoint partition P of B consisting of elements of $\mathcal{D}$. Since $\pi(x)$ is regular and $\pi(x) \leq 2^\omega <$ 1st w.i, $\pi(x)$ is a successor cardinal number. So, for every $x \in P$ there is a κ such that the algebra B_x is

(i) $(\kappa, \cdot, 2)$-distributive,

(ii) nowhere $(\kappa^+, \cdot, 2)$-distributive,

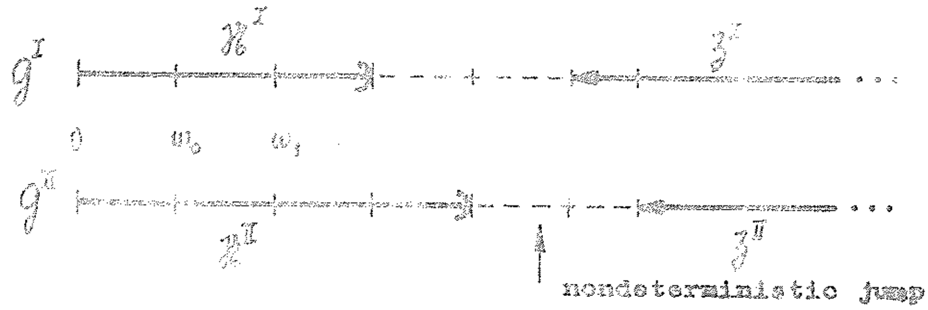
(iii) $d(B_x) \leq 2^\omega$,

(iv) Black wins $\mathcal{G}(B_x, \omega)$.

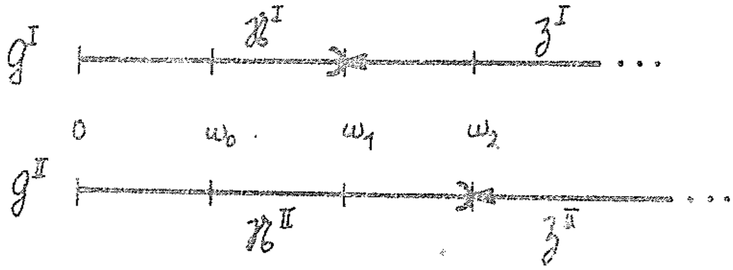
From Theorem 3 we get a σ-closed dense subset of algebra B_x for all $x \in P$ and, henceforth, also for the algebra B.

§ 3. Some game properties of Boolean algebras

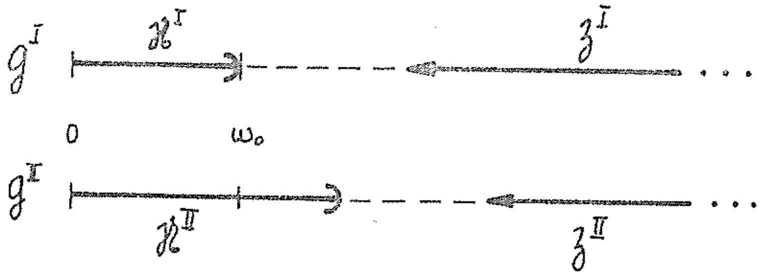
Investigations carried out in this final part are motivated by a result of C. Gray ([5]). Consider the games $\mathcal{G}^{I}$ and $\mathcal{G}^{II}$ as defined in § 1. Gray proved that there is a Boolean algebra E such that Black wins $\mathcal{G}^{II}(E, \omega_1)$ but does not win $\mathcal{G}^{I}(E, \omega_1)$ and therefore algebra E does not have an ω_2-closed dense subset. For sets $\mathcal{H}^{I}$, $\mathcal{H}^{II}$, $\mathfrak{Z}^{I}$, $\mathfrak{Z}^{II}$ defined in the introduction we easily get $\mathcal{H}^{I} \subseteq \mathcal{H}^{II}$ and $\mathfrak{Z}^{II} \subseteq \mathfrak{Z}^{I}$. To ilustrate the game properties of an algebra let us draw the following pictures



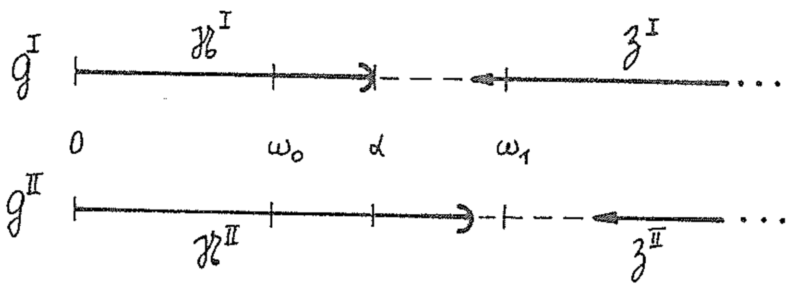
The diagram of the Gray's algebra B looks as follows



Since the games $g^{I}(B,\omega)$ and $g^{II}(B,\omega)$ are the same, the following situation



does not occur. Our next result shows that even for α such that $\omega < \alpha < \omega_1$ the Gray's trick cannot be used, i.e.



also cannot occur.

Theorem 6. Assume that B is a complete Boolean algebra. Then sup($\mathcal{K}^{I}(B)$) and sup($\mathcal{K}^{II}(B)$) are regular cardinal numbers.

Proof. In both cases it suffices to prove that if $\{\alpha_\xi : \xi < \kappa\} \subseteq \mathcal{K}^{\cdot}$ is a sequence of limit ordinal numbers and $\kappa \in \mathcal{K}^{\cdot}$ is a limit ordinal number, then $\sum_{\xi<\kappa} \alpha_\xi \in \mathcal{K}^{\cdot}$ (here $\sum$ denotes the ordinal sum).

(a) Let σ be a winning strategy for Black in $\mathcal{G}^{I}(B,\kappa)$ and σ_ξ, $\xi<\kappa$, are the winning strategies for Black in games $\mathcal{G}^{I}(B,\alpha_\xi)$. The desired strategy for the game $\mathcal{G}^{I}(B, \sum\alpha_\xi)$ we construct "interweaving σ with σ_ξ ". (See also picture).

For $\xi \leq \kappa$, put $\delta_\xi = \sum_{\eta<\xi} \alpha_\eta$ (e.g. $\delta_0 = 0$, $\delta_\kappa = \sum_{\xi<\kappa} \alpha_\xi$). We define H, the Black's strategy case by case: for $\xi<\kappa$ and $0 < \zeta < \alpha_\xi$ put (remember our convention about double indexes)

$$b(\delta_\xi+\zeta) = H(w_0,\dots,w(\delta_\xi),w(\delta_\xi+1),\dots,w(\delta_\xi+\zeta)) =$$
$$= \sigma_\xi(w(\delta_\xi+1),w(\delta_\xi+2),\dots,w(\delta_\xi+\zeta))$$

and $b(\delta_\xi) = H(w_0,w_1,w_2,\dots,w(\delta_\xi)) =$
$$= \sigma(w_0,w(\delta_1),w(\delta_2),\dots,w(\delta_\xi))$$

White	Black
w_0	
	$\sigma(w_0)$
w_1	
	$\sigma_0(w_1)$
w_2	
	$\sigma_0(w_1,w_2)$
⋮	

continued on the next page

White	Black
	continued
	$\vdots$
$w(\alpha_0)$	
	$\sigma(w_0, w(\alpha_0))$
$w(\alpha_0+1)$	
	$\sigma_1(w(\alpha_0+1))$
$w(\alpha_0+2)$	
	$\sigma_1(w(\alpha_0+1), w(\alpha_0+2))$
	$\vdots$
$w(\alpha_0+\alpha_1)$	
	$\sigma(w_0, w(\alpha_0), w(\alpha_0+\alpha_1))$
$w(\alpha_0+\alpha_1+1)$	
	$\sigma_2(w(\alpha_0+\alpha_1+1))$
	$\vdots$

For every $\xi < \kappa$ the sequence

$$w(\delta_\xi+1), b(\delta_\xi+1), \ldots, w(\delta_\xi+\zeta), b(\delta_\xi+\zeta), \ldots : \zeta < \alpha_\xi$$

is a play of $\mathcal{G}^{I}(B, \alpha_\xi)$ in which Black follows σ_ξ and hence $\bigwedge\{w_\eta : \eta < \delta_{\xi+1}\} \neq \mathbb{O}$, and so White can choose the $w(\delta_{\xi+1})$. Moreover, the sequence

$$w_0, b_0, w(\delta_1), b(\delta_1), \ldots, w(\delta_\xi), b(\delta_\xi), \ldots : \xi < \kappa$$

is a play of $\mathcal{G}^{I}(B, \kappa)$ in which Black uses σ and therefore $\bigwedge\{w_\alpha : \alpha < \delta_\kappa\} \neq \mathbb{O}$. Hence, the defined strategy H is winning for the player Black in the game $\mathcal{G}^{I}(B, \sum_{\xi<\kappa} \alpha_\xi)$.

(b) The proof for the game $\mathcal{G}^{II}$ is similar but we have to be more carefull on limit stages. Let $\alpha_\xi, \kappa, \sigma, \sigma_\xi, \delta_\xi$ be the same as in (a) but for g^{II}. Define the strategy H case by case:

$$b_0 = H(w_0) = \sigma(w_0)$$

for $0 \leq \xi < \omega$ and $1 \leq \zeta < \alpha_\xi$, respectively for $\omega < \xi < \kappa$, ξ nonlimit and $1 < \zeta < \alpha_\xi$, put

$$b(\delta_\xi) = H(w_0, w_1, \ldots, w_\alpha, \ldots : \alpha < \delta_\xi) \text{ is arbitrary}$$
$$\text{up to the conditions of the game}$$

$$\begin{aligned} b(\delta_\xi + 1) &= H(w_0, w_1, \ldots, w(\delta_\xi)) = \\ &= (w_0, w(\delta_1), w(\delta_2), \ldots, w(\delta_{\omega+1}), \ldots, w(\delta_\xi)) = \\ &= (w_0, w(\delta_1), \ldots, w(\delta_\eta), \ldots : \eta \leq \xi \ \& \ \eta \text{ nonlimit}) \end{aligned}$$

$$\begin{aligned} b(\delta_\xi + \zeta) &= H(w_0, w_1, \ldots, w(\delta_\xi + \vartheta), \ldots : \vartheta < \zeta) = \\ &= \sigma_\xi(w(\delta_\xi + 1), \ldots, w(\delta_\xi + \vartheta), \ldots : \vartheta < \zeta) \end{aligned}$$

for $\omega \leq \xi < \kappa$ and ξ limit and $0 < \zeta < \alpha_\xi$, put

$$\begin{aligned} b(\delta_\xi) &= H(w_0, w_1, \ldots, w_\alpha, \ldots : \alpha < \delta_\xi) = \\ &= \sigma(w_0, w(\delta_1), \ldots, w(\delta_\eta), \ldots : \eta < \xi \ \& \ \eta \text{ nonlimit}) \end{aligned}$$

$$\begin{aligned} b(\delta_\xi + \zeta) &= H(w_0, w_1, \ldots, w(\delta_\xi + \vartheta), \ldots : \vartheta < \zeta) = \\ &= \sigma_\xi(w(\delta_\xi), w(\delta_\xi + 1), \ldots, w(\delta_\xi + \vartheta), \ldots : \vartheta < \zeta). \end{aligned}$$

White	Black
w_0	
	$\sigma(w_0)$
w_1	
	$\sigma_0(w_1)$
w_2	
	$\sigma_0(w_1, w_2)$
$\vdots$	

continued on the
next page

White	Black
	⋮ continued
	$b(\delta_1)$ arbitrary
$w(\delta_1)$	
	$\sigma(w_0, w(\delta_1)) = b(\delta_1+1)$
$w(\delta_1+1)$	
	$\sigma_1(w(\delta_1+1)) = b(\delta_1+2)$
$w(\delta_1+2)$	
	$\sigma_1(w(\delta_1+1), w(\delta_1+2))$
	⋮
	$b(\delta_\omega) = \sigma(w_0, w(\delta_1), \ldots, w(\delta_n), \ldots : n < \omega)$
$w(\delta_\omega)$	
	$\sigma_\omega(w(\delta_\omega)) = b(\delta_\omega+1)$
$w(\delta_\omega+1)$	
	$\sigma_\omega(w(\delta_\omega), w(\delta_\omega+1)) = b(\delta_\omega+2)$
	⋮
	$b(\delta_{\omega+1})$ arbitrary
$w(\delta_{\omega+1})$	
	$\sigma(w_0, w(\delta_1), \ldots, w(\delta_n), \ldots, w(\delta_{\omega+1})) = b(\delta_{\omega+1}+1)$
$w(\delta_{\omega+1}+1)$	
	$\sigma_{\omega+1}(w(\delta_{\omega+1}+1)) = b(\delta_{\omega+1}+2)$
	⋮

It is easy to see that for $\xi < \omega$ or ξ nonlimit the sequence

$$w(\delta_\xi+1), b(\delta_\xi+1), \ldots, b(\delta_\xi+\omega), w(\delta_\xi+\omega), \ldots$$
$$\ldots, b(\delta_\xi+\vartheta), w(\delta_\xi+\vartheta), \ldots : \vartheta < \alpha_\xi$$

respectively for $\xi > \omega$ and ξ limit the sequence

$$w(\delta_\xi), b(\delta_\xi +1), w(\delta_\xi +1), b(\delta_\xi +2), \dots, b(\delta_\xi +\omega), w(\delta_\xi +\omega), \dots$$
$$\dots, b(\delta_\xi +\vartheta), w(\delta_\xi +\vartheta), \dots : \vartheta < \alpha_\xi$$

is a play of the game $\mathcal{G}^{II}(B, \alpha_\xi)$ in which Black follows σ_ξ. Moreover, the sequence

$$w_0, b_0, w(\delta_1), b(\delta_1+1), \dots, w(\delta_n), b(\delta_n+1), \dots$$
$$\dots, b(\delta_\omega), w(\delta_{\omega+1}), b(\delta_{\omega+1}+1), w(\delta_{\omega+2}), \dots$$
$$\dots, b(\delta_\xi), w(\delta_{\xi+1}), b(\delta_{\xi+1}+1), w(\delta_{\xi+2}), \dots : \xi < \kappa$$

is a play of $\mathcal{G}^{II}(B, \kappa)$ and $\bigwedge\{w_\alpha : \alpha < \delta_\kappa\} = \bigwedge\{w(\delta_\xi) : \xi < \kappa\} \neq \mathbb{O}$. Hence H is a winning strategy for the Black in $\mathcal{G}^{II}(B, \sum_{\xi<\kappa} \alpha_\xi)$.

<u>Remark 3</u>. As a corollary of the previous theorem we get

$$\kappa \in \mathcal{H}^{I} \longrightarrow \kappa^+ \subseteq \mathcal{H}^{I}$$
$$\kappa \in \mathcal{H}^{II} \longrightarrow \kappa^+ \subseteq \mathcal{H}^{II}$$

If the algebra B has a σ-closed dense subset, then $\omega_1 \subseteq \mathcal{H}^{I}(B) \cap \mathcal{H}^{II}(B)$. But from Theorem 6 we get that

$(\mathcal{H}^{II}(B) - \mathcal{H}^{I}(B)) \cap \omega_1 = \emptyset$ for every Boolean algebra B. So the nonexistence of a σ-closed dense subset cannot be proved using differences between $\mathcal{H}^{I}$ and $\mathcal{H}^{II}$.

Now let us recall the notion of a simultaneous strategy introduced in [10]. It is our feeling, that the existence of the simultaneous strategy can be a loophole for breaking down the Jech's problem.

<u>Definition</u>. We say that the player Black has a simultaneous winning strategy in the game $\mathcal{G}^{I}$ on algebra B ($\mathcal{G}^{II}$, respectively) if there is one strategy

$$\sigma : \bigcup\{{}^{\beta}B ; \beta < \omega_1\} \longrightarrow B$$

such that σ is winning for Black in each game $\mathcal{G}^{I}(B, \alpha)$ for

$\alpha < \omega_1$ ($\mathcal{G}^{II}(B,\alpha)$ for $\alpha < \omega_1$, respectively).

Note that if B has a σ-closed dense subset, then there is a simultaneous strategy for $\mathcal{G}^{I}$ and $\mathcal{G}^{II}$, and if Black has simultaneous winning strategy in the game $\mathcal{G}^{I}$ then he has simultaneous winning strategy also in the game $\mathcal{G}^{II}$. Hence the following problems arise:

Problem 1. Does "Black has a winning strategy in $\mathcal{G}(B,\omega)$" imply "There is a simultaneous winning strategy for Black in the game $\mathcal{G}^{II}$ on B" ?

2. Does the existence of simultaneous winning strategy in the game $\mathcal{G}^{I}$ imply the existence of simultaneous winning strategy in the game $\mathcal{G}^{II}$?

In [4] is proved that min($\mathfrak{z}^{II}(B)$) is the smallest cardinal number δ for which the algebra B is not $(\delta,\cdot,2)$-distributive. For the Gray's algebra E we have min($\mathfrak{z}^{I}(E)$) < min($\mathfrak{z}^{II}(E)$) Our last theorem answers the remaining question. It shows (together with the previous results) that $\mathcal{G}^{I}$ is realy a "cardinal-type" game.

Theorem 7. For every Boolean algebra B, min($\mathfrak{z}^{I}(B)$) is a regular cardinal number.

Proof. Assume that $\{\alpha_\xi : \xi < \kappa\}$ is a sequence of limit ordinal numbers disjoint with $\mathfrak{z}^{I}(B)$ and let $\kappa \notin \mathfrak{z}^{I}(B)$ is a limit ordinal number. We show that $\sum_{\xi<\kappa} \alpha_\xi \notin \mathfrak{z}^{I}(B)$.

Assume on the contrary that σ is a winning strategy for the White in the game $\mathcal{G}^{I}(B, \sum \alpha_\xi)$. First we introduce some notation. Put $\delta_\xi = \sum_{\eta<\xi} \alpha_\eta$. For a sequence $\bar{b} = (b_0, b_1, \dots, b_\eta, \dots : \eta < \delta_\xi)$ define $\sigma[\bar{b}]$ a strategy for the White in the game

$\mathcal{G}^{I}(B, \alpha_\xi)$ as follows

$\sigma[\bar{b}](y_0, y_1, \dots, y_\mu, \dots : \mu < \vartheta < \alpha_\xi) =$

$= \sigma(b_0, b_1, \dots, b_\eta, \dots, y_0, y_1, y_2, \dots, y_\mu, \dots : \eta < \alpha_\xi \ \& \ \mu < \vartheta)$

Assume that ρ is some strategy for the White in the game $\mathcal{G}^{I}(B, \alpha_\xi)$. As $\alpha_\xi \notin \mathfrak{Z}^{I}(B)$, there is a sequence $\{b_0, b_1, \dots, b_\eta, \dots : \eta < \alpha_\xi\}$ of entries of Black such that the following sequence $\rho(\phi), b_0, \rho(b_0), b_1, \rho(b_0, b_1), b_2, \dots, \rho(b_0, b_1, \dots, b_\mu, \dots : \mu < \eta),$
$, b_\eta, \dots : \eta < \alpha_\xi$
is a play of $\mathcal{G}^{I}(B, \alpha_\xi)$ which White does not win. Using AC, for each ρ fix one such a sequence and denote it $\bar{b}[\rho]$. That is, for every ρ there is a sequence $\bar{b}[\rho]$ of Black's entries such that ρ fails against this play. To show that σ fails in $\mathcal{G}^{I}(B, \sum \alpha_\xi)$, we shall iterate these two operations (e.g. $\rho[\bar{b}[\sigma]]$). For two (or more) sequences $b_0, b_1, \dots$ the concatenation is denoted $b_0 {}^\frown b_1$ (respectively $b_0 {}^\frown b_1 {}^\frown \dots$).

Construction. There is a strategy H such that for every descending sequence $\{x_\alpha : \alpha < \kappa\}$ of nonzero elements of the algebra B there are sequences $\{\bar{b}_\xi : \xi < \kappa\}$ and $\{\sigma_\xi : \xi < \kappa\}$ such that

(1) $\bar{b}_\xi = \{b^\xi_\eta : \eta < \alpha_\xi\}$ is a descending sequence of nonzero elements of algebra B,

(2) σ_ξ is a strategy for White in the game $\mathcal{G}^{I}(B, \alpha_\xi)$ and H is a strategy for White in the game $\mathcal{G}^{I}(B, \kappa)$

(3) $\sigma_0 = \sigma[\{x_0\}]$

(4) $\bar{b}_0 = \bar{b}[\sigma_0] = \bar{b}[\sigma[\{x_0\}]]$

(5) $\sigma_\xi = \sigma[\{x_0\} {}^\frown \bar{b}_0 {}^\frown \{x_1\} {}^\frown \bar{b}_1 {}^\frown \dots {}^\frown \bar{b}_\eta {}^\frown \{x_{\eta+1}\} {}^\frown \dots {}^\frown \{x_\xi\} : \eta < \xi]$

(6) $\bar{b}_\xi = \bar{b}[\sigma_\xi]$ depends only on σ and $\{x_\eta ; \eta \le \xi\}$

(7) $H(\phi) = \sigma(\phi)$

$H(x_0, x_1, \dots, x_\eta, \dots : \eta < \xi < \kappa) =$

$= \sigma(\{x_0\} {}^\frown \bar{b}_0 {}^\frown \{x_1\} {}^\frown \bar{b}_1 {}^\frown \dots {}^\frown \{x_\eta\} {}^\frown \bar{b}_\eta {}^\frown \dots : \eta < \kappa).$

	White	Black	
	$H(\phi) = \sigma(\phi)$		
		x_0	
	$\sigma_0(\phi) = \sigma(x_0)$		
		b_0^0	
$\sigma_0 = \sigma[\{x_0\}]$	$\sigma_0(b_0^0) = \sigma(x_0, b_0^0)$		$\bar{b}_0 = \bar{b}[\sigma_0]$
		b_1^0	
		$\vdots$	
	$H(x_0) = \sigma(\{x_0\}^\frown \bar{b}_0)$		
		x_1	
	$\sigma_1(\phi) = \sigma(\{x_0\}^\frown \bar{b}_0{}^\frown \{x_1\})$		
		b_0^1	
σ_1	$\sigma_1(b_0^1) = \sigma(\{x_0\}^\frown \bar{b}_0{}^\frown \{x_1\}^\frown \{b_0^1\})$		$\bar{b}_1$
		b_1^1	
		$\vdots$	
	$H(x_0, \ldots, x_n, \ldots : n < \omega) =$ $= \sigma(\{x_0\}^\frown \bar{b}_0{}^\frown \ldots {}^\frown \{x_n\}^\frown \bar{b}_n{}^\frown \ldots : n < \omega)$		
		x_ω	
	$\sigma_\omega(\phi)$		
σ_ω		b_0^ω	$\bar{b}_\omega$
$\vdots$		$\vdots$	$\vdots$

The construction is straightforward. Note that H is a strategy for White in $\mathcal{G}^{I}(B, \kappa)$. The value of $H(\{x_\eta : \eta < \xi < \kappa\})$

depends only on the sequence $\{x_\alpha : \alpha < \kappa\}$. But $\kappa \notin \mathfrak{Z}^{I}(B)$. Then $\bar{y} = \bar{b}[H] = \{y_\xi : \xi < \kappa\}$ is a sequence such that $H(\phi), y_0, H(y_0), y_1, \ldots, H(y_0, \ldots, y_\eta, \ldots : \eta < \xi), y_\xi, \ldots : \xi < \kappa$ is a play of $\mathcal{G}^{I}(B, \kappa)$ and $\bigwedge_{\xi<\kappa} y_\xi \neq \mathbb{O}$.

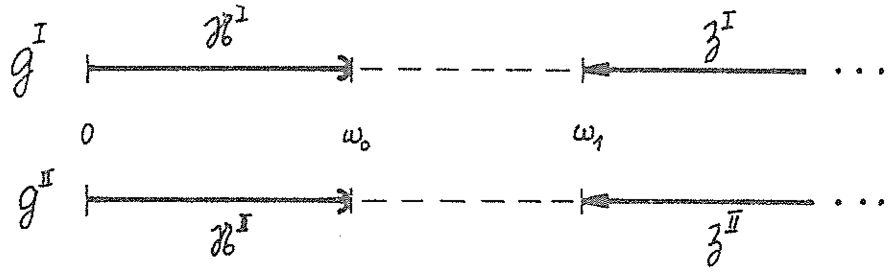
From the construction (for this particular $\bar{y}$) there is a sequence $\{\bar{b}_\xi : \xi < \kappa\}$ such that the sequence

$\{y_0\}^\frown \bar{b}_0{}^\frown \{y_1\}^\frown \bar{b}_1{}^\frown \ldots {}^\frown \{y_\xi\}^\frown \bar{b}_\xi{}^\frown \ldots : \xi < \kappa$ is a sequence of Black's entries in the game $\mathcal{G}^{I}(B, \sum_{\xi<\kappa} \alpha_\xi)$ in which White uses σ. Contradiction.

<u>Remark 4</u>. If B is a Souslin algebra or the nondetermined algebra constructed by T.Jech in [6], then

$$\sup \mathcal{H}^{I}(B) = \sup \mathcal{H}^{II}(B) = \omega \quad \text{and}$$
$$\min \mathfrak{Z}^{I}(B) = \min \mathfrak{Z}^{II}(B) = \omega_1.$$

The diagram of B looks like



<u>Remark 5</u> (Problems). We dont know whether for each algebra B is

$$|(\mathcal{H}^{II}(B) - \mathcal{H}^{I}(B)) \cap \mathrm{Card}| \leq 1 \quad \text{and}$$
$$|(\mathfrak{Z}^{I}(B) - \mathfrak{Z}^{II}(B)) \cap \mathrm{Card}| \leq 1.$$

A.Hajnal asked whether

$$|(\mathfrak{Z}^{I}(B) \cap \mathcal{H}^{II}(B)) \cap \mathrm{Card}| \leq 1$$

is true.

Is there an algebra B such that $\min(\mathfrak{Z}^{II}(B)) = \aleph_{\omega+1}$ and $\min(\mathfrak{Z}^{I}(B)) < \min(\mathfrak{Z}^{II}(B))$ (and hence $\min(\mathfrak{Z}^{I}(B)) = \aleph_n$ for some $n < \omega$?

Does for any four regular cardinal numbers $\nu_1, \nu_2, \eta_1, \eta_2$ such that

$$\begin{array}{ccccc} & \leq & \eta_1 & \leq & \\ \nu_1 & & & & \eta_2 \\ & \leq & \nu_2 & \leq & \end{array}$$

exist an Boolean algebra such that $\sup(\mathcal{H}^{I}(B)) = \nu_1$, $\sup(\mathcal{H}^{II}(B)) = \nu_2$, $\min(\mathfrak{Z}^{I}(B)) = \eta_1$ and $\min(\mathfrak{Z}^{II}(B)) = \eta_2$?

The last couple of problems concerns the algebra $B_0 = \mathcal{P}(\omega)_{/fin}$. We define

$$\kappa_c = \min\{|F| \;;\; F \subseteq B_0^{+} \text{ is centered and } \bigwedge F = 0\}$$

$$\kappa_t = \min\{|T| \;;\; T \subseteq B_0^{+} \text{ is a tower and } \bigwedge T = 0\}$$

$$\kappa = \min\{\sigma \;;\; B_0 \text{ is not } (\sigma,\cdot,2)\text{-distributive}\}$$

$$\nu_1 = \sup(\mathcal{H}^{I}(B_0)) \qquad \nu_2 = \sup(\mathcal{H}^{II}(B_0))$$

$$\eta_1 = \min(\mathfrak{Z}^{I}(B_0)).$$

Easily $\eta_2 = \min(\mathfrak{Z}^{II}(B_0)) = \kappa$.

Parameters κ_c, κ_t and κ were investigated by B.Balcar, J.Pelant and P.Simon in [1]. P.L.Dordal ([3]) proved Con(ZFC + $\kappa_c < \kappa$). A.Szymanski and H.X.Zhou proved in [9] that $\kappa_c = \omega_1$ implies $\kappa_t = \omega_1$. This result together with Dordal´s metatheorem gives Con($\kappa_t < \kappa$ + ZFC). The following diagram shows the relations between these characteristics of the algebra $\mathcal{P}(\omega)_{/fin}$.

$$\begin{array}{ccccccccc} & & & & & \leq & \eta_1 & \leq & \\ \kappa_c & \leq & \kappa_t & \leq & \nu_1 & & & & \kappa = \eta_2 \\ & & & & & \leq & \nu_2 & \leq & \end{array}$$

We ask: Which inequalities can be consistently sharp?

Prove or disprove Con(ZFC + $\eta_1 < \nu_2$).

Is there a simultaneous winning strategy for Black in the game $\mathcal{G}^{I}$ on $\mathcal{P}(\omega)_{/fin}$ of the length ν_1 ($\mathcal{G}^{II}$ and ν_2 respectively)?

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