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A NEW SUFFICIENT CONDITION FOR THE EXISTENCE OF Q-POINTS IN $\beta\omega - \omega$

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ABSTRACT

We prove that if the minimal size of a dominating family of functions from ω to ω is strictly smaller than the Baire number of $\omega^* = \beta\omega - \omega$, then there are Q-points in ω^* (also called rare ultrafilters). We also discuss several related results.

INTRODUCTION

Throughout the paper a standard set-theoretical notation is used, see T. Jech [6]. Thus, ω is the set of natural numbers, $[\omega]^\omega$ is the set of all infinite subsets of ω, $[\omega]^\omega$ is the set of all finite subsets of ω, ${}^\omega\omega$ is the set of all functions from ω to ω. We shall study various types of uniform ultrafilters on ω, i.e. points of the remainder ω^* of the Čech–Stone compactification of ω. Let $j \in \omega^*$. We say that

$$j \text{ is a } \textit{P-point} \text{ if } \forall \{U_n : n \in \omega\} \subseteq [\omega]^\omega,\ U_n \notin j\ \exists X \in j$$
$$\forall n\ |U_n \cap X| < \aleph_0;$$

j is *selective* (also *Ramsey*) if $\forall \{U_n\colon n \in \omega\} \subseteq [\omega]^\omega$,

$$U_n \notin j \ \exists X \in j \ \forall n \ |U_n \cap X| \leqslant 1;$$

j is a *Q-point* (also *rare*) if $\forall \{U_n\colon n \in \omega\} \subseteq [\omega]^{<\omega}$

$$\exists X \in j \ \forall n \ |U_n \cap X| \leqslant 1.$$

For $x \in [\omega]^\omega$ denote $e(x) \in {}^\omega\omega$ the enumeration of x (i.e. the unique increasing function from ω onto x). Then

j is *rapid* (also *semi-Q-point*) if the set $\{e(x)\colon x \in j\}$ is a dominating family in ${}^\omega\omega$;

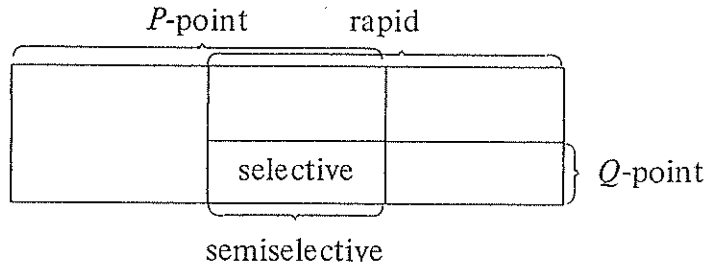
j is *semiselective* if j is both P-point and rapid.

Recall the following facts:

j is selective iff it is both P-point and Q-point;

j is a Q-point implies j is rapid.

The mutual relationship among the sets of these points is described by the following picture.



Historically, the interest in seeking sufficient conditions guaranteeing the existence of various types of points in ω^* preceded the negative results – it is consistent that no such points exist. Below we list both the positive (ordered according to the strength of the corresponding sufficient condition) and negative results.

G. Mokobodzki [13]: CH $\longrightarrow$ $\exists$ rapid.

D. Booth [2]: CH, MA, MA (σ-centered) $\longrightarrow$ $\exists$ selective.

A.R.D. Mathias, A. Taylor [11]: $\Delta = \omega_1 \longrightarrow \exists\, Q$-point.

(In fact, the proof gives $\kappa_t = \Delta \to \exists\, Q$-point.) Note that none of these two conditions implies the existence of selective ufs. Recall that

$$\Delta = \min \{|D|:\ D \subseteq {}^{\omega}\omega \text{ and } D \text{ is a dominating family in } {}^{\omega}\omega\},$$

$$\kappa_t = \min \{|T|:\ T \subseteq [\omega]^{\omega} \text{ and } T \text{ is a nowhere dense tower}\}$$

(see [1], [5]).

B. Balcar – J. Pelant – P. Simon [1]: $n(\omega^*) > 2^{\omega} \to \exists$ selective, where

$$n(\omega^*) = \min \{|\mathscr{F}|:\ \mathscr{F} \text{ is a covering of } \omega^* \text{ by nowhere dense sets}\}$$

the Baire number of ω^*.

J. Ketonen [8]: Cov (K) $= 2^{\omega} \longrightarrow \exists$ selective, where

$$\text{Cov (K)} = \min \{|\mathscr{I}|:\ \mathscr{I} \text{ is a covering of } [0, 1] \text{ by sets of 1-st category}\}.$$

K. Kunen [9]: Con (ZFC + there are no selective ufs.).

S. Shelah [16]: Con (ZFC + there are no P-points).

A.W. Miller [12]: Con (ZFC + there are no rapid ufs.).

THE MAIN RESULT

In an earlier version of our paper we dealt with rapid filters and the proof of the original statement was quite tricky. The proof of the stronger result presented here is based on the same idea (communicated to us by P. Simon) by which the result of Mathias and Taylor can be proved (cf. [11]).

Theorem. *Assume that the smallest size of dominating family of functions from* ${}^{\omega}\omega$ *is strictly smaller than the Baire number of* $\omega^* = \beta\omega - \omega$. *Then there are* Q*-points in* ω^*. *In symbols,*

$$\Delta < n(\omega^*) \longrightarrow \exists\, Q\text{-point}.$$

Proof. To construct a Q-point we have to take care of all partitions of ω consisting of finite sets. Our first step is to show that every such partition $\mathscr{R}$ can be replaced by an "interval partition" – determined by a function $f_{\mathscr{R}}$. Next we show that if suffices to take care of Δ-many interval partitions, which are connected with iterations of functions from a dominating family.

Suppose that $\mathscr{R} = \{R_n\colon n \in \omega\} \subseteq [\omega]^{<\omega}$ is a partition of ω consisting of finite nonempty sets and $0 \in R_0$. Define a function $f_{\mathscr{R}} \in {}^{\omega}\omega$ and a sequence $\{n_i\colon 0 < i \in \omega\}$ as follows

$$f_{\mathscr{R}}(0) = \max\{\bigcup\{X \in \mathscr{R}\colon X \cap (R_0 \cup [0, 2]) \neq \phi\}\}$$

and for $0 < i \in \omega$ put

$$n_i = \min\{n\colon R_n \cap [0, f_{\mathscr{R}}(i-1)] = \phi\},$$

$$f_{\mathscr{R}}(i) = \max\{\bigcup\{X \in \mathscr{R}\colon X \cap [0, \max(R_{n_i}) + 2] \neq \phi\}\}.$$

Note that if $Z \subseteq \omega$ is a selector for the partition $\{[f_{\mathscr{R}}(i), f_{\mathscr{R}}(i+1))\colon i \in \omega\}$, then

$$(\forall X \in \mathscr{R})(|X \cap Z| \leqslant 2).$$

Claim. *Suppose that f and g are strictly increasing functions greater than identity $+ 1$ and $f < g$. Define*

$$\bar{g}(0) = g(0)$$

$$\bar{g}(n+1) = g(\bar{g}(n)) \quad \textit{for} \quad 0 < n \in \omega.$$

Then $f(\xi) \leqslant \bar{g}(n) < f(\xi + 1)$ *implies* $\bar{g}(n+1) \geqslant f(\xi + 1)$, *for each* $\xi, n \in \omega$.

Proof. Assume the contrary and take n, ξ such that

$$f(\xi + 1) > \bar{g}(n+1) > \bar{g}(n) \geqslant f(\xi).$$

Immediately,

$$f(\xi + 1) > \bar{g}(n+1) = g(\bar{g}(n)) > f(\bar{g}(n))$$

and $\xi + 1 < f(\xi) \leqslant \bar{g}(n)$ gives $f(\bar{g}(n)) > f(\xi + 1)$. A contradiction.

We continue now the proof of the theorem. Observe that if $f_{\mathscr{A}} < g$, then for every selector Z of the partition $\{[\bar{g}(n), \bar{g}(n+1)) \colon n \in \omega\}$ we have

$$(\forall X \in \mathscr{A})(|X \cap Z| \leqslant 4).$$

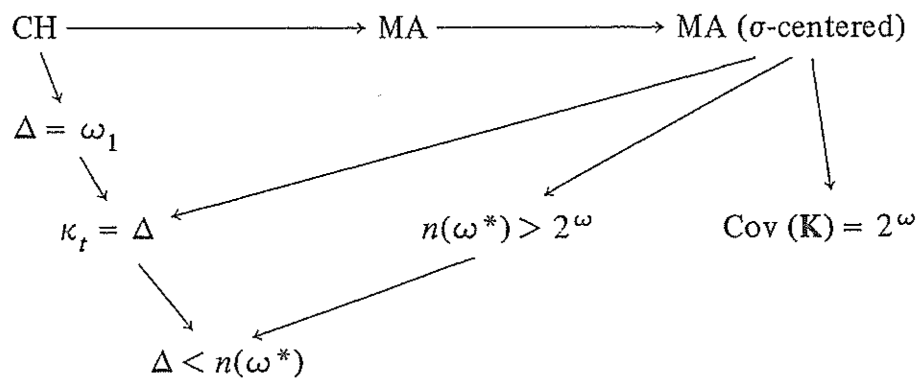
Let $\mathscr{D} = \{g_\alpha \colon \alpha < \Delta\}$ be a dominating family in ${}^\omega\omega$. For each $\alpha < \Delta$ put

$$U_\alpha = \{A \in [\omega]^\omega \colon (\forall n)(|(\omega - A) \cap [\bar{g}_\alpha(n), \bar{g}_\alpha(n+1))| \leqslant 1)\}.$$

Then $\mathscr{U}_\alpha = \{p \in \omega^* \colon p \supseteq U_\alpha\}$ is a nowhere dense set. As $\Delta < n(\omega^*)$, there is a $p \in \omega^* - \bigcup\{\mathscr{U}_\alpha \colon \alpha < \Delta\}$. This is the required Q-point (rare ultrafilter).

DISCUSSION

The picture bellow shows how the conditions mentioned in the historical outline and our condition are interrelated.



Our condition is more general than all previous – except the Ketonen's, the two beeing incomparable; even in every model for $\Delta < n(\omega^*) \leqslant 2^\omega$ we have Con (K) $< 2^\omega$.

The consistency. We do not need to construct any new model. To establish

$$\text{Con}\,(\text{ZFC} + \omega_1 = \kappa_t < \omega_3 = \Delta < n(\omega^*) = \omega_4 = 2^\omega),$$

we use a result of P. L. Dordal [5]. We start with a model $\mathscr{M}_0$ of GCH. By a c.c.c-extension R we obtain $\mathscr{M}_1$ such that

$$\mathcal{M}_1 \models 2^\omega = \omega_4 \quad \text{and} \quad \omega_4^{\omega_3} = \omega_4 .$$

Then using c.c.c.-ordering Q from Lemma 2.25 in [5], p. 62, constructed in $\mathcal{M}_1$, we extend it to $\mathcal{M}_2$ and

$$\mathcal{M}_2 \models \mathrm{MA}_{<\omega_3} \quad \text{and} \quad \kappa_t = \Delta = \omega_3 \quad \text{and} \quad 2^\omega = \omega_4 .$$

Note that using a technique, which has been explained e.g. in the paper of L. Bukovský [3], the same effect as using $R * Q$ can be obtained. Namely, first add ω_4 Cohen reals and then ω_3 times iterate $(\mathrm{ST}_{\omega_3} * \mathrm{C} * P_{\omega_\omega})$ in a c.c.c. manner (ST denotes the Solovay–Tennenbaum closure ([3]), C is the Cohen algebra and P_{ω_ω} adds a dominating family).

Extending $\mathcal{M}_2$ by $P = \left(\bigcup_{\alpha < \omega_1} {}^{\alpha}2 \right) \cap \mathcal{M}_0$ we obtain the desired model $\mathcal{M}_3$. Dordal in [5], Theorem 2.1, p. 31 proved that forcing with P has the following effect:

$$\mathcal{M}_3 \models \kappa_t = \omega_1 \quad \text{and} \quad \Delta = \omega_3 \quad \text{and} \quad 2^\omega = \omega_4 .$$

As $\Delta = \omega_3 < 2^\omega$ and $\mathrm{MA}_{<\omega_3}$ holds in $\mathcal{M}_2$, using results of [1] we get $\mathcal{M}_2 \models n(\omega^*) = \omega_4$. As P remains countably distributive in $\mathcal{M}_2$ and $|P| = \aleph_1$, we can use Theorem 3.12 of [5], p. 78 to prove that in $\mathcal{M}_3$ no new maximal towers of size ω_2 do occur (in $\mathcal{M}_2$ there were no such towers because of $\kappa_t = \omega_3$). Consequently, no shattering matrix with all branches of size at most ω_2 does occur in $\mathcal{M}_3$ and, according to [1], this means that

$$\mathcal{M}_3 \models n(\omega^*) = \omega_4 .$$

We do not know of any such model with $2^\omega = \omega_3$ nor even with $\kappa_t^+ = \Delta$.

Remark 1. In both Shelah's model for no P-points ([16]) and Kunen's model for no selective ultrafilters ([9]) there is $\Delta = \omega_1$ and therefore $\Delta < n(\omega^*)$. So our result cannot be extended to selective ultrafilters.

Remark 2. $\Delta < n(\omega^*)$ implies the existence of a scale. The only known (to us) condition which implies the existence of selective ultrafilters and does not imply the existence of a scale is the Ketonen's one. While writing the last version of this paper a publication of a relevant paper [7] has been announced.

Remark 3. In Laver's model for Borel conjecture [10] there are no rapid ultrafilters (Miller [12]) and hence we have $\Delta = n(\omega^*) = \omega_2$. We do not know whether in this model the nondistributivity of the algebra $\mathscr{P}(\omega)$ / fin is equal to ω_1 or ω_2. The case with ω_2 would be of some interest.

MIXED TYPES OF POINTS OF ω^*

We list some results which guarantee the existence of points in ω^* showing that the inclusions among various sets of points in ω^* mentioned in the introduction are proper (see the picture).

D. Booth [2]: MA $\longrightarrow$ $(\exists j)$ (j is both P-point and not Q-point).

K. Kunen [9]: MA $\longrightarrow$ $(\exists j)$ (j is both P-point and not rapid) and $(\exists j)$ (j is both semiselective and not selective).

We note that these three implications hold also under MA (σ-centered).

L. Bukovský – E. Copláková [4] (the title of [4] is a bit misleading):

Con ($\neg$ MA (σ-centered) and $(\exists j)$ (j is both semiselective and not selective)).

Stronger results of this type say (loosely speaking) "if one box is nonempty, then also some other box is nonempty".

A.W. Miller [12]: If there is a rapid ultrafilter then there is also a rapid ultrafilter which is neither P-point nor Q-point.

Proposition. *If there is a selective ultrafilter and* $\Delta < \chi$, *then there is a* Q*-point which is not selective.* (χ denotes the minimal possible character of a point in ω^*.)

We do not know of any set-theoretical condition which would follow from the existence of any of the mentioned types of points.

THE STRUCTURE OF MEASURE ON THE REAL LINE VERSUS THE STRUCTURE OF POWER SET ALGEBRA

This final section differs from the previous ones. Consider the following correspondence: each infinite subset of ω uniquely determines (and it is determined by) a real number from the unit interval (via the characteristic function, up to diadic expression of reals), in symbols

$$j \subseteq \mathscr{P}(\omega) \leftrightsquigarrow j \subseteq [0, 1].$$

Let us recall two results concerning this correspondence.

W. Sierpiński [14]: j is an ultrafilter $\longrightarrow j$ is nonmeasurable.

M. Talagrand [15]: $\mathscr{F}$ is a rapid filter $\longrightarrow$ $\mathscr{F}$ is nonmeasurable.

A striking counterpart to these theorems is the following.

Observation. *All uniform filters (ultrafilters) have a measurable base (of measure zero).*

Really, according to the structure of the power-set algebra (or $\beta\omega - \omega$), all properties of a filter are given by its base but the measurability of a base does not imply the measurability of the whole filter. Is there anything deeper behind this phenomenon?

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