SIMULTANEOUS STRATEGIES
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The paper is devoted to the study of the existence of a λ-closed dense subsets of a Boolean algebra under certain game--theoretical properties. T.Jech in [3] introduced the following game. Let B be a complete Boolean algebra and α an ordinal number. The transfinite game $\mathcal{G}(B,\alpha)$ is played between two players White and Black. Let White and Black define a decreasing sequence

$$w_0 \geqslant b_1 \geqslant w_2 \geqslant b_3 \geqslant \ldots \geqslant w_{\beta+2n} \geqslant b_{\beta+2n+1} \geqslant \ldots \qquad (1)$$

of nonzero elements of B of the length $\leq \alpha$ by taking turns defining its entries. (White chooses $w_{\beta+2n}$, for β limit $< \alpha$, $n \in \omega$; Black chooses $b_{\beta+2n+1}$) The play is won by Black if the sequence (1) has nonzero intersection and length α, and by White if the intersection is $\mathcal{O}$. A winning strategy for Black in the game $\mathcal{G}(B,\alpha)$ is a function $\mathfrak{S} : \bigcup\{{}^{\beta}B : \beta < \alpha\} \longrightarrow B$ with the property that Black wins every play (1) in which he follows $\mathfrak{S}$.

Theorem 1 (T.Jech [3]). Assume B is a complete Boolean algebra and λ an uncountable cardinal number. Then $(a) \rightarrow (c) \rightarrow (d)$, where

(a) Algebra B has a λ-closed dense subset.

(c) Black has a winning strategy in the game $\mathcal{G}(B,\tau)$ for each $\tau < \lambda$.

(d) Algebra B is $(\tau, \infty, 2)$ distributive for each $\tau < \lambda$.

Basically, our research was motivated by the question whether $(c) \rightarrow (a)$, i.e. whether the existence of winning strategy for Black implies that B has a λ-closed dense subset. The following defini-tion concerns the structure of the set of all strategies and is powerfull also for limit cardinal numbers.

Definition 2. We say that Black has a simultaneous winning strate-gy of the length λ in the algebra B if there is one strategy $\mathfrak{S} : \bigcup\{{}^{\alpha}B : \alpha < \lambda\} \longrightarrow B$ such that $\mathfrak{S}$ is winning for Black in each game $\mathcal{G}(B,\alpha)$ for $\alpha < \lambda$.

Lemma 3. $(a) \rightarrow (b) \rightarrow (c)$, where

(a) and (c) are as in the Theorem 1, and

(b) Black has simultaneous strategy for B of the length λ .

PROOF. Is obvious.

In what follows we describe a type of algebras for which implication (b)$\rightarrow$(a) holds, namely the ones which has a tree-base. We give some conditions under which a boolean algebra has a tree--base. As a consequence we get an characterization of the algebra Col (λ,κ). At the end of this paper we give some historical comments.

§1. Notations, definitions, constructions. Let B be a complete Boolean algebra, $\leq$ is the canonical ordering of the algebra B, λ,κ,τ are cardinal and α,β,γ ordinal numbers, Lim denotes the class of limit ordinal numbers. $B^+ = B - \{0\}$, $B\restriction u$ is a partial algebra, hsat(B) denotes the hereditarily saturatedness. By P and Q we denote a maximal partition of B, system $\mathcal{H} = \{P_\alpha : \alpha<\lambda\}$ is called a matrix, P_α's are columns of $\mathcal{H}$, $x \in P_\alpha$ is an element of the matrix $\mathcal{H}$, $P << Q$ denotes that P refines Q, $P << \mathcal{H}$ if P refines each P_α. $\mathcal{H}$ is said to be monotone provided $\alpha<\beta$ implies $P_\beta << P_\alpha$. Remark, that if $\mathcal{H}$ is monotone, then $(\cup\mathcal{H}, \leq)$ forms a tree and P_α is the α-th level of this tree. Let $\Omega = \{Q_\alpha : \alpha<\lambda\}$. Then Ω refines $\mathcal{H}$ if each Q_α refines P_α. For $x\in B^+$, $x\wedge\wedge P =\{y\wedge x : y\in P_\alpha\}\cap B^+$. The algebra B is said to be (λ,∞,κ)-distributive provided for any matrix $\mathcal{H} = \{P_\alpha:\alpha<\lambda\}$ there is a maximal partition P of B such that $(\forall x\in P)(\forall\alpha<\lambda)(|x\wedge\wedge P_\alpha| < \lambda)$. Algebra B is called (λ,∞,κ)-nowhere distributive if for each $x\in B^+$ the algebra $B\restriction x$ is not (λ,∞,κ)-distributive. Recall that B is (λ,∞,κ)-nowhere distri-butive iff there is a matrix $\mathcal{H} = \{P_\alpha : \alpha<\lambda\}$ such that for each $x\in B^+$ there is some $\alpha<\lambda$ with $|x\wedge\wedge P_\alpha| \geq \kappa$. In this case we say that $\mathcal{H}$ is a matrix witnessing to (λ,∞,κ)-nowhere distributivity and if B is $(\tau,\infty,2)$-distributive for all $\tau<\lambda$ (e.g. if B has si-multaneous winning strategy for Black of length λ) then $\mathcal{H}$ will be assumed to be monotone. We say that $D\subseteq B^+$ is a λ-closed dense subset of algebra B (we say sometimes base instead of dense sub-set) if $(\forall x\in B^+)(\exists y\in D)(y\leq x)$ and for every decreasing sequence $\{a_\alpha : \alpha<\tau\}\subseteq D$ of the length $\tau<\lambda$ there is an $y\in D$ such that $y\leq a_\alpha$ for each $\alpha<\tau$. d(B) denotes the density of B. A matrix $\mathcal{H}= \{P_\alpha : \alpha<\lambda\}$ is said to be a tree-base of the algebra B of the length λ if $\cup\mathcal{H}$ is a base (i.e. elements of $\mathcal{H}$ form a base) and $\mathcal{H}$ is monotone (i.e. $(\cup\mathcal{H},\leq)$ is a tree). For unexplained notation we refer to [2].

§2. Tree-base, game-tree and simultaneous strategy. The idea to construct a λ-closed dense subset of a Boolean algebra from a

tree appeared independently in [1] and [4]. We develope the tech-
nics of [4] to obtain more general results.

Theorem 4. Assume B is a complete Boolean algebra which has a
tree-base of the length λ and B has simultaneous winning strategy
for Black of length λ . Then B has a λ -closed dense subset.

PROOF. The idea of the proof is following. We introduce a natural
notion of a game tree according to strategy σ for Black in which
every branch is a play of game $\mathcal{G}$ in which Black follows σ and
the elements of the tree split on even levels (turns of the White)
and odd levels are determined by even ones and σ . Then by trans-
finite induction we construct an game-tree according to σ which
refines the base $\mathcal{H}$ and this is the desired λ -closed base.

Let $\mathcal{H} = \{P_\alpha : \alpha < \lambda\}$ be a tree-base of B and σ is a simul-
taneous winning strategy. A tree $(T, \le)$ is called a game-tree of
length λ according to the strategy σ if

(i) $T \subseteq B^+$, $\le$ is the canonical ordering of B, T has length λ

(ii) $(\forall \alpha \in \lambda \cap \text{Lim})(\forall n \in \omega)T_{\alpha+2n+1}$ is a maximal partition of B
(where T_β denotes the β -th level of the tree T)

(iii) $(\forall x \in T_{\alpha+2n+1})(x = \sigma(\text{pr}(x)))$ (where pr(x) is the sequence
of predecessors of x in the tree T).

Note that any branch of T is a play of game $\mathcal{G}$ in which Black
follows σ , so it is of the length λ . So if T is a base of algebra
B it is a λ -closed base. To this end we have to construct a game-
-tree T which refines the tree base $\mathcal{H}$ i.e.

(iv) $(\forall \alpha \in \lambda \cap \text{Lim})(\forall n \in \omega)(T_{\alpha+2n+1} \ll P_{\alpha+n})$
Assume that $T_{<\alpha+2n}$ is constructed already. For $T_{\alpha+2n}$ take the maxi-
mal system such that

(v) $T_{\alpha+2n}$ refines $T_{<\alpha+2n}$

(vi) $\{\sigma(\text{pr}(x),x) : x \in T_{\alpha+2n}\}$ is a maximal partition of B
which refines $P_{\alpha+n}$.

This is possible because $T_{<\alpha+2n}$ is a game tree of the length
$\alpha+2n$ according to the simultaneous strategy σ of the length $\lambda>\alpha+2n$,
algebra B is complete and $(\alpha,\infty,2)$ distributive.

<div align="right">q.e.d.</div>

Now we consider the question, when does there exists a tree-
-base for a Boolean algebra. We need the following technical lemma.

Lemma 5. Assume λ is an uncountable cardinal number, B is a comple-
te Boolean algebra which has a simultaneous winning strategy for
Black of length λ and $\mathcal{H} = \{P_\alpha : \alpha < \lambda\}$ is a monotone matrix
wittnessing to (λ,∞,κ)-nowhere distributivity of B. Then for each
$x \in B^+$ the following holds

(i) $|\{(y,\alpha) : y \in P_\alpha \& x \wedge y \neq \mathbb{O}\}| \geq \kappa^\partial$ (i.e. x intersects many elements of the matrix $\textcircled{H}$)

and (ii) $\mathrm{hsat}(B) > \kappa^\partial$

PROOF. The main idea again is to construct something like a game tree of type $({}^\partial\kappa, \subseteq)$, but its relation to the matrix $\textcircled{H}$ is a little bit more complicated as in the Theorem 4.

Assume w.l.o.g. that x = 1 . In fact for (i) we have to prove that $|\cup\textcircled{H}| \geq \kappa^\partial$. The existence of simultaneous strategy of length ∂ implies $(\tau,\infty,2)$-distributivity for $\tau < \partial$ and B is (∂,∞,κ)-nowhere distributive - we have that ∂ is regular. By transfinite induction we construct F and T such that

(i) $T \subseteq \cup\textcircled{H}$ is a Boolean tree (i.e. (i) of Theorem 4 holds)

(ii) $F : \cup\{{}^\alpha\kappa : \alpha < \partial\} \to T$ is an tree isomorphism

(iii) for each $f \in {}^\partial\kappa$ the sequence $F(f \restriction 0) = w_0$, $\mathfrak{S}(w_0) = b_1$, $w_2 = b_1 \wedge F(f \restriction 1)$, $b_3 = \mathfrak{S}(w_2),\ldots,w_{\alpha+2n} = \bigwedge\{b_\beta : \beta < \alpha+2n\} \wedge F(f\restriction\alpha+2n)$, $b_{\alpha+2n+1} = \mathfrak{S}(w_{\alpha+2n}),\ldots$ is a play in which Black follows $\mathfrak{S}$ - the simultaneous strategy of the length ∂ .

As $T \subseteq \cup\textcircled{H}$ and $\textcircled{H}$ is monotone and ∂ is regular we have immediately (i). For $\alpha < \partial$ T_α is a disjoint system, so $\mathrm{hsat}(B) \geq \kappa^\partial$. If $\kappa^\partial > \partial$ then κ^∂ is a singular cardinal number and so $\mathrm{hsat}(B) > \kappa^\partial$. If $\kappa^\partial = \partial$ then using $\mathfrak{S}$ we can construct a strictly decreasing tower $\{t_\alpha : \alpha < \partial\}$ and then $\{t_\alpha - t_{\alpha+1} : \alpha < \partial\}$ is the desired disjoint system of power $\kappa^\partial = \partial$.

                                                                      q.e.d.

Theorem 6. Let B be a (∂,∞,κ)-nowhere distributive complete Boolean algebra with density $d(B) = \kappa^\partial$ for which the Black has simultaneous winning strategy of the length ∂ . Then B has a tree base of length ∂ .

PROOF. Let $D \subseteq B^+$ be a base of size κ^∂. As each element of D intersects κ^∂ elements of $\Omega = \{Q_\alpha : \alpha < \partial\}$ - a monotone matrix witnessing to (∂,∞,κ)-nowhere distributivity - there is a one to one mapping $f : D \to \cup\Omega$ such that $x \wedge f(x) \neq \mathbb{O}$. Split each $y \in \mathrm{rng}(f)$ into two elements $y \wedge f^{-1}(y)$ and $y - f^{-1}(y)$. We obtain a matrix Ω' which refines D , i.e. $\cup\Omega'$ is a base. Using $(\tau,\infty,2)$ distributivity for each $\tau < \partial$ we obtain a monotone matrix $\textcircled{H}$ - the desired tree-base.

                                                                      q.e.d.

Corollary 7. Let B be a (∂,∞,κ)-nowhere distributive complete Boolean algebra with density $d(B) = \kappa^\partial$ which has a simultaneous winning strategy for Black of length ∂ . Then

$$B \cong \mathrm{Col}(\partial,\kappa^\partial) = \mathrm{Col}(\partial,\kappa) .$$

PROOF. From Lemma 5 we have hsat(B) $= (\kappa^{\partial})^{+}$ so in the construction
of the Theorem 4 we can add the condition

$$(\forall x \in T_{\alpha+2n+1})(|\{y \in T_{\alpha+2n+2} : y \leq x\}| \geq \kappa^{\partial}) .$$

The base we obtain is isomorphic to the base of the algebra
$Col(\lambda, \kappa^{\partial})$.

<div align="right">q.e.d.</div>

We have seen that the existence of a tree base is important.
Observe that if B has a tree base of the length λ and $\kappa <$ hsat(B)
then B is $(\lambda, \infty, \kappa)$-nowhere distributive. About inverse implication
we know only what Theorem 6 says.

The case $\lambda = \omega_1$ was studied in [4]. This paper is a genera-
lization of the results of [4] for $\lambda = \omega_1$. For nonlimit cardinal
numbers M. Foreman in [1] proved that if $\kappa = \lambda^{+}$, $\lambda^{\sigma} = \lambda$, B is
$(\lambda, \infty, 2)$-distributive, d(B) $= \lambda^{+}$ and Black wins $\mathcal{G}(B, \gamma)$ then
B has a γ^{+}-closed dense subset. Our technics works without cardi-
nal assumptions and also for limit cardinals, but needs stronger
strategical assumption.

The notion of the simultaneous strategy gives a view into
the structure of all strategies. The problem, whether (c) $\rightarrow$ (a),
((a), (b), (c) are as in Theorem 1 and Lemma 2) can be splited
into two: whether (c) $\rightarrow$ (b) and (b) $\rightarrow$ (a).

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