General Topology and its Relations
to Modern Analysis and Algebra V
Proc. Fifth Prague Topol. Symp. 1981
J. Novak (ed.)

657 – 662

A TRANSFINITE BOOLEAN GAME AND A GENERALIZATION OF KRIPKE'S EMBEDDING THEOREM

P. Vojtáš

Abstract: Our talk is devoted to a Boolean game introduced by T.Jech. We define a new game-theoretic property of Boolean algebras (namely, the existence of a simultaneous winning strategy for Black) and give a new characterization of the algebra $Col(\lambda_1, \kappa)$. The result is related to a problem of T.Jech on strategies of Boolean games and yields a generalization of Kripke's embedding theorem. At the end we formulate two problems concerning the structure of the set of all strategies.

AMS(MOS) subject classification (1970): 04 A 20
Key words: Games on Boolean algebras, collapsing algebra.

§ 1. Introduction. The aim of this paper is to introduce a new game-theoretical property of Boolean algebras and present some results related to a problem posed by T.Jech about games on Boolean algebras. For historical comments we refer to the introductory part in [2] and the concluding remarks in [4]. Recall some facts from [4]. Let B be a complete Boolean algebra and $\alpha < \omega_1$ an ordinal number. Consider the following transfinite game $\mathcal{G}(B,\alpha)$ between two players White and Black. Let White and Black define a decreasing sequence

$$(1) \qquad w_0 \geq b_0 \geq w_1 \geq b_1 \geq \dots \geq w_\beta \geq b_\beta \geq \dots$$

of nonzero elements of B of length $\leq \alpha$ by taking turns defining its entries. The play is won by Black if the sequence (1) has nonzero intersection and length α, and by White if the intersection is 0.

Topological reformulation of this game as well as the results can be obtained by passing to the Stone space of the given algebra, i.e., entries are open sets and winning condition reads: the intersection has or has not a nonempty interior.

A winning strategy for Black is a function $\sigma : \bigcup_{\beta<\alpha} {}^{\beta}B \longrightarrow B$

with the property that Black wins every play (1) in which he follows σ. We say that algebra B has

- property (G), if Black has a winning strategy in the game $\mathcal{G}(B,\omega)$;
- property (G'), if Black has a winning positional strategy, i.e., a strategy $\sigma: B \to B$ which depends only the last move of the oponent

T h e o r e m 1. (T.Jech,[4]). (a)$\to$(b)$\to$(d)$\to$(e), where

a) B has $\aleph_0$-closed dense subset;

b) B has property (G');

d) B has property (G);

e) B is $(\aleph_0,\infty,2)$ distributive.

Basically, our research was motivated by Problem 6 in [5]: Show that "B has property (G)" does not imply "B has $\aleph_0$-closed dense subset".

T h e o r e m 2. If B has property (G), then for each $\alpha<\omega_1$ Black has a winning strategy σ_α in the game $\mathcal{G}(B,\alpha)$.

Proof. By induction. Easily, if Black wins $\mathcal{G}(B,\alpha)$ and $\mathcal{G}(B,\beta)$ then he wins $\mathcal{G}(B,\alpha+\beta)$, too. Suppose σ_n is winning in $\mathcal{G}(B,\alpha_n)$ and σ in $\mathcal{G}(B,\omega)$. Similarly, if Black follows σ on steps $\sum_{i=0}^{m}\alpha_i$ and follows σ_n on steps between $(\sum_{i=0}^{m}\alpha_i)+1$ and $\sum_{i=0}^{m+1}\alpha_i$, then he wins every play of $\mathcal{G}(B,\sum_{n=0}^{\infty}\alpha_n)$.

Of course, for different α, β the strategies may differ. This observation leads us to the following definition.

D e f i n i t i o n 3. We say that Black has a simultaneous winning strategy (or B has property (S), or B is an (S)-algebra) if there is one strategy $\sigma: \bigcup_{\beta<\omega_1} {}^{\beta}B \to B$ such that σ is winning for Black in each game $\mathcal{G}(B,\alpha)$ for $\alpha<\omega_1$.

Basic information about (S)-algebras is provided by the following lemmas.

L e m m a 4. (b)$\to$(c)$\to$(d), where (b), (d) are as in Theorem 1 and (c) B has property (S).

Proof. Obvious.

L e m m a 5. Every (S)-algebra is $\aleph_1$-representable.

Proof. T.Jech in [4] proved that every (G')-algebra is $\aleph_1$-representable. Our statement can be proved in the same way.

In § 3 we present more results concerning property (S) and find a certain type of algebras for which implication (c)$\to$(a) holds.

ACKNOWLEDGEMENT. The author would like to express his gratitude to B. Balcar who called his attention to Boolean games.

§ 2. Notations and definitions. Let $\kappa \geq 2$ be a cardinal number and B a complete Boolean algebra. We shall use the following symbols: $d(B)$ denotes the density of B, $B^+ = B - \{\mathbb{O}\}$, $B \upharpoonright u$ is a partial algebra, $hsat(B)$ denotes the hereditarily saturatedness. By P, Q, W we usualy denote a maximal partition of B, system $\Theta = \{P_\alpha ; \alpha < \omega_1\}$ is called a matrix, P_α's are column's of Θ, $x \in P_\alpha$ is an element of matrix Θ, $P << Q$ denotes that P refines Q, $P << \Theta$ if P refines each P_α, Θ is said to be monotone provided $\alpha < \beta$ implies $P_\beta << P_\alpha$. Remark that if Θ is monotone, then $(\bigcup\Theta, \leq)$ forms a tree. Let $\Omega = \{Q_\alpha ; \alpha < \omega_1\}$. Then Ω refines Θ if each Q_α refines P_α. For $x \in B^+$, $x \wedge\wedge P = \{y \wedge x;\ y \in P_\alpha\} \cap B^+$. The algebra B is said to be $(\aleph_1, \infty, \kappa)$-distributive (see [9]) provided for any matrix $\Theta = \{P_\alpha ; \alpha < \omega_1\}$ there is a maximal partition P of B such that $(\forall x \in P)(\forall \alpha < \omega_1)(|x \wedge\wedge P_\alpha| < \kappa)$, and B is called $(\aleph_1, \infty, \kappa)$-nowhere distributive if for each $x \in B^+$ the algebra $B \upharpoonright x$ is not $(\aleph_1, \infty, \kappa)$-distributive. Recall that B is $(\aleph_1, \infty, \kappa)$-nowhere distributive iff there is a matrix $\Theta = \{P_\alpha ; \alpha < \omega_1\}$ such that for each $x \in B^+$ there is some $\alpha < \omega_1$ with $|x \wedge\wedge P_\alpha| \geq \kappa$. In this case we say that Θ is a matrix witnessing to $(\aleph_1, \infty, \kappa)$-nowhere distributivity. If B is $(\aleph_0, \infty, 2)$-distributive (e.g., if B is an (S)-algebra), Θ will be assumed to be monotone. Denote $C(\aleph_1, \kappa) = \{f;\ f$ is a mapping from $\alpha < \omega_1$ into $\kappa\} = \bigcup_{\alpha < \omega_1} {}^{\alpha}\kappa$ and put $f \geq g$ if $f \subseteq g$. The (unique) complete Boolean algebra containing $C(\aleph_1, \kappa)$ as a dense subset is denoted by $Col(\aleph_1, \kappa)$, (see [3]). Recall that the Stone space of the algebra $Col(\aleph_1, \kappa)$ is the $\aleph_1$-product of $\aleph_1$ copies of κ equipped with the discrete topology. For unexplained notation terminology we refer to [1] and [3].

§ 3. Simultaneous strategy

L e m m a 6. Assume B is a (G)-algebra and Ω is a monotone matrix witnessing to $(\aleph_1, \infty, \kappa)$-nowhere distributivity of B. Then for each $x \in B^+$ there is a $P \subseteq \bigcup\Omega$ such that

(i) P is a system of mutually disjoint elements;

(ii) $|P| = \kappa^{\aleph_0}$;

(iii) $(\forall y \in P)\ (x \wedge y \neq \mathbb{O})$.

Proof. W.l.o.g. assume that $x = \mathbb{1}$. By transfinite induction we construct a mapping $F : \bigcup_{n<\omega} {}^n\kappa \to \bigcup\Omega$ such that

(1) $\operatorname{rng}(F)$ is a subtree of the tree $(\bigcup\Omega, \leq)$;

(2) F is an tree isomorphism;

(3) $(\forall f \in {}^\omega\kappa)\ (\bigwedge_{n<\omega} F(f(n)) \neq \mathbb{O})$.

Let σ be a winning strategy for Black. In the construction of F it is necessary to check that for each $f \in {}^\omega\kappa$ the sequence

$$F(f(0)) = w_0,\ \sigma(w_0) = b_1,\ \ w_1 = b_0 \wedge F(f(1)),\ \ b_1 = \sigma(w_1),\ \ldots$$

is a play in which Black follows σ . From $(\aleph_1,\infty,\kappa)$-nowhere distributivity we have that for any b_0 there is an $\alpha < \omega_1$ with $|b_0 \wedge \wedge Q_\alpha| \geq \kappa$. Put F an injection of the set $\{g; g \in {}^2\kappa \,\&\, g(0) = f(0)\}$ into $\{y \in Q_\alpha;\ b_0 \wedge y \neq \mathbb{O}\}$.

The rest of the proof is now straight forward. For any $f \in {}^\omega\kappa$ take $x_f \in \bigcup\Omega$ such that $x_f \wedge \bigwedge_{n<\omega} F(f(n)) \neq \mathbb{O}$. Then $\{x_f; f \in {}^\omega\kappa\}$ is the desired system.

The next theorem gives necessary conditions for a algebra B to be isomorphic with the algebra Col $(\aleph_1, \kappa^{\aleph_0})$ (see § 2).

T h e o r e m 7. Assume B is an $(\aleph_1,\infty,\kappa)$-nowhere distributive complete Boolean algebra, the density $d(B) = \kappa^{\aleph_0}$, and B has property (S) . Then

$$B \cong \operatorname{Col}\,(\aleph_1, \kappa^{\aleph_0}) .$$

Proof. We show that there is a dense subset $D \subseteq B^+$ such that $(D, \leq)$ and $C(\aleph_1, \kappa^{\aleph_0}) = \bigcup_{\alpha<\omega_1} {}^\alpha(\kappa^{\aleph_0})$ are order isomorphic. The result follows then from the fact that two complete Boolean algebras with isomorphic dense subsets are isomorphic (cf. [7]). We look for D of the form $D = \bigcup\Theta$. Let $E \subseteq B^+$ be a dense subset with $|E| = \kappa^{\aleph_0}$ and let $\Omega = \{Q_\alpha; \alpha < \omega_1\}$ be a matrix from Lemma 6. As each element of E intersects $\kappa^{\aleph_0}$ elements of Ω , there is an $f : E \xrightarrow{1-1} \bigcup\Omega$ such that $f(x) \wedge x \neq \mathbb{O}$. Put $Q'_\alpha = \{y;\ (y \in Q_\alpha \,\&\, y \notin \operatorname{rng}(f)) \vee (\exists x \in E)(y = x \wedge f(x) \vee y = f(x) - x)\}$. Put $\Omega' = \{Q'_\alpha, \alpha < \omega_1\}$. Then $\bigcup\Omega'$ is a dense subset of B . Every matrix which refines Ω' forms a dense subset, too. W.l.o.g. we assume that Ω' is monotone. A tree $(T, \leq)$ is said to be a game tree provided $T \subseteq B^+$, $\leq$ is the canonical ordering of B and branches of T are plays of game $\mathcal{G}$. Denote by $w_{\alpha+n}$ the elements of odd level $\ell_{\alpha+2n}$, and denote by $b_{\alpha+n}$ the elements of even $\ell_{\alpha+2n+1}$. Let σ be a simultaneous strategy for Black. By transfinite induction we construct a game tree $(T, \leq)$ such that:

1) Every branch of T is a play of $\mathcal{G}$ in which Black follows σ;

2) The system of even levels $\mathcal{H} = \{\ell_{\alpha+2n+1};\ \alpha \text{ limit},\ n<\omega\}$ (choices of Black) is a matrix refining $\mathcal{Q}'$;

3) for each $b_{\alpha+n} \in \ell_{\alpha+2n+1}$ there are κ-many successornodes in the level $\ell_{\alpha+2(n+1)}$ (and also in $\ell_{\alpha+2n+3}$).

Assume that $T_{\leq\alpha+2n+1}$ is already constructed. Pick $b\in\ell_{\alpha+2n+1}$. Let $pr(b)$ is the sequence of predecessors of b in T. Let $s_b \subseteq B\upharpoonright b$ be a maximal system such that

a) $q_b = \{\sigma(\langle pr(b), b, x\rangle);\ x\in s_b\}$ is a maximal partition of $B\upharpoonright b$;

b) $|q_b| = \kappa^{\aleph_0}$;

c) q_b refines $b \wedge\wedge Q'_{\alpha+n+1}$.

As hsat $(B) > \kappa^{\aleph_0}$ (from (ii) of Lemma 6), take s'_b a disjoint system of power $\kappa^{\aleph_0}$ refining $b\wedge\wedge Q'_{\alpha+n+1}$. Let s_b be maximal such that q_b is disjoint and refines $b\wedge\wedge Q'_{\alpha+n+1}$. Assume that $\bigvee q_b < b$. Take $y\in b\wedge\wedge Q'_{\alpha+n+1}$ such that $y - \bigvee q_b \neq \mathbb{O}$ and put $s^* = s_b \cup \{y - \bigvee q_b\}$. Then s^* fulfills a)b)c) – a contradiction with maximality of s_b. We define

$$\ell_{\alpha+2n+2} = \bigcup\{s_b;\ b\in\ell_{\alpha+2n+1}\}.$$

For a limit α put $\ell_\alpha = \{\bigwedge v;\ v \text{ is a branch of } T_{<\alpha}\}$. As σ is a simultaneous strategy for Black, ℓ_α is well defined for all $\alpha<\omega_1$! The system of "nonlimit" even levels $\mathcal{H} = \{\ell_{\alpha+2n+1};\ \alpha \text{ limit},\ n\geq 1\}$ forms a dense subset of B which is isomorphic to $C(\aleph_1, \kappa^{\aleph_0})$.

Note that Theorem 7 is a strengthening of the corresponding proposition in [8], where instead of the property (S) it is assumed that B has an $\aleph_0$-closed dense subset (see [1]).

The next corollary sheds some light on the cited problem of T. Jech.

Corollary 8. Assume B is $(\aleph_1,\infty,\kappa)$-nowhere distributive and $d(B) = \kappa^{\aleph_0}$. Then

$$(S) \longrightarrow B \text{ has an } \aleph_0\text{-closed dense subset}.$$

Proof. Obvious.

Corollary 9. Assume B has the property (S) and $d(B) \leq \kappa^{\aleph_0}$. Then there is a complete embedding of B into the algebra $\mathrm{Col}\,(\aleph_1, \kappa^{\aleph_0})$.

Proof. Take $B\times \mathrm{Col}\,(\aleph_1,\kappa^{\aleph_0})$. This algebra fulfills conditions of Theorem 7. Thus we have $B \hookrightarrow B\times \mathrm{Col}\,(\aleph_1,\kappa^{\aleph_0}) \cong$ $\cong \mathrm{Col}\,(\aleph_1, \kappa^{\aleph_0})$.

In [6] it is proved: for every B there is a complete embedding into the algebra Col $(\aleph_0, |B|)$. The assertion of Corollary 9 was proved in [8] under a stronger assumption that B has $\aleph_0$-closed dense subset (see [1]).

Problem. a) Does have every (S)-algebra an $\aleph_0$-closed dense subset ?

b) Does (G) imply (S) ?

References

[1] B.Balcar, P.Simon, P.Vojtáš: Refinement properties and extensions of filters in Boolean algebras, to appear in Trans. Amer. Math. Soc.

[2] F.Galvin, T.Jech, M.Magidor: An ideal game. J. Symb. Logic 43, 2(1978), 284-292.

[3] T.Jech: Set theory, Academic Press, New York, 1978.

[4] T.Jech: A game theoretic property of Boolean algebras, in Logic Colloquium 1977 (A.Mac Intyre et al., eds.), 135-144, North-Holland Publ. Co., Amsterdam 1978.

[5] T.Jech: More game theoretic properties of Boolean algebras, preprint.

[6] S.Kripke: An extension of a theorem of Gaifman-Hales-Solovay, Fund. Math. 61(1967), 29-32.

[7] R.Sikorski: Boolean algebras, Springer Verlag, 1964.

[8] P.Vojtáš: Refining systems on Boolean algebras, doctoral disertation, (in Slovak), Charles University, Prague 1980.

[9] P.Vopěnka, P.Hájek: The theory of semisets, Academia Prague and North Halland Amsterdam, 1972.

Mathematical Institute of the Slovak Academy of Sciences
Karpatská 5, 040 01 Košice
Czechoslovakia