BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences mathématiques
Vol. XXVIII, No. 11—12, 1980

Refinement and Properties and Extending of Filters

by

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Presented by C. RYLL-NARDZEWSKI on September 6, 1979

Summary. We announce some results concerning the disjoint refinement property in Boolean algebras with emphasis on families of subsets of a given set modulo some ideal. Questions of Comfort and Hindman, Woods and Hušek and van Douwen are answered. No proof is given here.

Recall the classical theorem due to Bernstein, Sierpiński, Kuratowski. Let κ be an infinite cardinal and let $\{a_\alpha : \alpha \in \kappa\}$ be the system of sets with $|a_\alpha| = \kappa$ for each $\alpha \in \kappa$. Then there exists a system $\{b_\alpha . \alpha \in \kappa\}$ such that each b_α is a set of cardinality κ, $b_\alpha \subset a_\alpha$ for each $\alpha \in \kappa$ and if $\alpha \neq \beta$ then $b_\alpha \cap b_\beta = \emptyset$, [18]. In the present paper we want to give a brief summary of results and problems more or less related to this theorem. Full proofs and further details will appear elsewhere.

We shall study the refinement properties of subsets of Boolean algebras. The Boolean algebraic approach and language seems to be convenient, though the main interest is devoted to family of subsets of the given set modulo some ideal. In what follows, all Boolean algebras are assumed to be infinite. The notation used throughout the paper is the usual one, see e.g. [10]. Let us start with the basic notion.

1. DEFINITION. Let κ be an infinite cardinal, $\mathscr{B}$ a Boolean algebra, $\mathscr{A} = \{a_\alpha : \alpha \in \kappa\}$ system of non-zero members of $\mathscr{B}$. The system $\mathscr{A}$ has a disjoint refinement, if there exists a set $\mathscr{D} = \{d_\alpha : \alpha \in \kappa\}$ of non-zero elements of $\mathscr{B}$ such that $d_\alpha \leqslant a_\alpha$ for each $\alpha \in \kappa$ and $d_\alpha \wedge d_\beta = 0$ whenever $\alpha \neq \beta$.

2. DEFINITION. Let $\mathscr{B}$ be a Boolean algebra. Saturatedness of $\mathscr{B}$ is

$$\mathrm{sat}\,(\mathscr{B}) = \min\,\{\lambda : \text{if } p \text{ is a partition of } \mathscr{B}, \text{ then } |p| < \lambda\}.$$

(A partition of $\mathscr{B}$ is a pairwise disjoint collection of non-zero elements of $\mathscr{B}$). Hereditary saturatedness of $\mathscr{B}$ is

$$\mathrm{hsat}\,(\mathscr{B}) = \min\,\{\mathrm{sat}\,(\mathscr{B}_x) : x \in \mathscr{B} - \{0\}\}.$$

Here $\mathscr{B}_x$ stands for the Boolean algebra of all $y \in \mathscr{B}$; $y \leqslant x$ with induced operations.

3. THEOREM [6]. *Let $\mathscr{B}$ be a Boolean algebra, κ an infinite cardinal. If* $\mathrm{hsat}\,(\mathscr{B}) > \kappa^+$, *then each system $\mathscr{A} \subset \mathscr{B} - \{0\}$ with $|\mathscr{A}| = \kappa$ has a disjoint refinement.*

4. Remarks. (a) Let κ be a regular infinite cardinal, i_F Fréchet ideal on κ, i.e. $i_F=\{M\subset\kappa: |M|<\kappa\}$. Then hsat $(\mathscr{P}(\kappa)/i_F)>\kappa^+$. Thus the previous theorem implies immediately Sierpiński's theorem mentioned at the beginning.

(b) The theorem is the best possible. The assumption hsat $(\mathscr{B})>\kappa^+$ cannot be weakened to hsat $(\mathscr{B})=\kappa^+$ without adding some restrictions to $\mathscr{B}$ or to $\mathscr{A}$.

(c) A particular case of Theorem 3 was proved by Baumgartner, Hajnal and Máté [4].

The most studied case of the existence of a disjoint refinement is the one with $\mathscr{A}$ closed under finite meets, in particular $\mathscr{A}$ an ultrafilter on $\mathscr{B}$. This question was studied mainly for $\mathscr{B}=\mathscr{P}(\kappa)/i_F$. Let us remember that ultrafilters on $\mathscr{P}(\kappa)/i_F$ correspond to uniform ultrafilters on κ, or to the points of $\mathscr{U}(\kappa)$. (For the other more general aspects concerning the disjoint refinement property of $\mathscr{P}(\kappa)$ modulo some ideal see the remarkable paper of Taylor [21].) The following is known:

If κ is an infinite cardinal and $2^\kappa=\kappa^+$, then each ultrafilter on $\mathscr{P}(\kappa)/i_F$ has a disjoint refinement ($\kappa=\omega$ Hindman [11], regular $\kappa>\omega$ Prikry [19], singular $\kappa>\omega$ & GCH Prikry [20]).

Without any additional assumptions, each ultrafilter on $\mathscr{P}(\omega)/i_F$ has a disjoint refinement, [7].

It was proved by Comfort and Hindman [9] that the existence of the disjoint refinement for the ultrafilter (in all the cases above) is equivalent to the following topological statement: Each point $p\in\mathscr{U}(\kappa)$ (=the space of all uniform ultrafilters on κ=the Stone space of $\mathscr{P}(\kappa)/i_F$) is a 2^κ-point. (That means there are 2^κ pairwise disjoint open sets such that each of them has the point P in this boundary.) It is not possible to prove without some additional set-theoretical assumptions that each point $p\in\mathscr{U}(\kappa)$ is a 2^κ-point, not even in the case $\kappa=\omega_1$ according to Baumartner's result [1]. (There need not be enough disjoint open sets in $\mathscr{U}(\omega_1)$.) Nevertheless we may ask whether each point in $\mathscr{U}(\kappa)$ is a κ^+-point. (One of the problems of Comfort and Hindman.) This leads naturally to the following definition.

5. Definition. Let $\mathscr{B}$ be a Boolean algebra, $\mathscr{A}\subset\mathscr{B}-\{0\}$, λ cardinal number. $\mathscr{A}$ is called to be a λ-collection if there is a family $\{p_\alpha:\alpha\in\lambda\}$ of partitions of $\mathscr{B}$ with the following properties:

(a) $\bigcup\{p_\alpha:\alpha\in\lambda\}$ is a partition of $\mathscr{B}$,

(b) $p_\alpha\cap p_{\alpha'}=\emptyset$ if $\alpha\neq\alpha'$.

(c) For each $a\in\mathscr{A}$ and for each $\alpha<\lambda$ there is an $x\in p_\alpha$ with $a\wedge x\neq 0$.

6. Theorem. (ZFC). *Let κ be an infinite regular cardinal. Then each filter $\mathscr{U}$ on $\mathscr{P}(\kappa)/i_F$ is a κ^+-collection.*

7. Corollary. *For regular uncountable cardinal κ, each point $p\in\mathscr{U}(\kappa)$ is a κ^+-point.*

8. Remark. In fact, more is true, κ^+ may be replaced by inf $\{|H|: H\subset{}^\kappa\kappa$ and H has no upper bound$\}$, where the upper bound of H is meant with respect to the order $\leqslant$ of ${}^\kappa\kappa$ defined by $f\leqslant g$ iff $|\{\xi\in\kappa: f(\xi)>g(\xi)\}|<\kappa$. (Compare with [3]).

Secondly, $\mathscr{U}$ need not be a filter. $\mathscr{U}$ may be e.g. the set of all stationary subsets of κ.

Now let us return to the problem whether a family closed under finite meets has a disjoint refinement. It turns out that if the algebra in question is complete, the existence of a disjoint refinement depends on the possible cardinality of a set of generators of an ultrafilter.

9. THEOREM. *Let $\mathscr{B}$ be a Boolean algebra, κ uncountable cardinal, let $\kappa^\kappa=\kappa$ and suppose that $\mathscr{B}$ has a dense subset of cardinality κ. Then there exists an ultrafilter on $\mathscr{B}$ generated by not more than κ elements of $\mathscr{B}$.*

(Notice that there is always an ultrafilter on $\mathscr{B}$ generated by not less than $|\mathscr{B}|$ members, provided that $\mathscr{B}$ is complete [2].)

10. COROLLARY. *Under* (CH), *there is an ultrafilter on Compl* $(\mathscr{P}(\omega)/i_F)$ (=*the completion of* $\mathscr{P}(\omega)/i_F$) *as well as on Col* (ω,ω_1) (≐*the complete Boolean algebra the dense set of which is the family of all functions from ω to ω_1 with a finite domain*) *generated by ω_1 members. Hence under* (CH) *there is a family $\mathscr{A}$ closed under finite intersection in Col* (ω,ω_1), $|\mathscr{A}|=\omega_1$, *having no disjoint refinement.*

11. REMARKS. (a) A similar result as in Corollary 10 was obtained independently by R. Laver under the assumption $\diamondsuit_{\omega_1}$, [16].

(b) Let an infinite cardinal κ be given. Does there exist a complete Boolean algebra $\mathscr{B}$ with hsat $(\mathscr{B})=\kappa^+$ containing a dense set of cardinality κ, such that each family $\mathscr{A}\subset\mathscr{B}-\{0\}$, $|\mathscr{A}|=\kappa$, closed under finite meets, has a disjoint refinement? This question was posed in [6]. According to Theorem 9, no such algebra exists if $\kappa^\kappa=\kappa>\omega$; the answer is yes for singular κ.

(c) The set-theoretical assumptions cannot be omitted from Theorem 9 and Corollary 10. This is illustrated by the following fact.

12. THEOREM. *Consider the algebra Col* (ω,ω_1). *If there is no ω_1-scale in ${}^\omega\omega$, then each family $\mathscr{A}\subset$ Col (ω,ω_1) closed under finite meets with $|\mathscr{A}|=\omega_1$ has a disjoint refinement.*

13. PROPOSITION. *Suppose* (CH), *let i be a nontrivial σ-complete ideal on ω_1. Then the following are equivalent:*

(a) *Each family $\mathscr{A}\subset\mathscr{P}(\omega_1)/i-\{0\}$ with $|\mathscr{A}|=\omega_1$ has a disjoint refinement.*

(b) *Each family $\mathscr{A}$ closed under finite meets, $\mathscr{A}\subset\mathscr{P}(\omega_1)/i-\{0\}$, $|\mathscr{A}|=\omega_1$ has a disjoint refinement.*

(c) *There is no family $\{X_\alpha:\alpha<\omega_1\}\subset\mathscr{P}(\omega_1)$ such that the ideal generated by $i\cup\{X_\alpha:\alpha<\omega_1\}$ is maximal.*

14. COROLLARY. *Suppose* (CH). *There exists a family $\{A_\alpha:\alpha<\omega_1\}$ of stationary subsets of ω_1 having no disjoint refinement by stationary sets (= the negation of Fodor's conjecture) if and only if there is a family $\{X_\alpha:\alpha<\omega_1\}$ such that all closed unbounded subsets of ω_1 together with $\{X_\alpha:\alpha<\omega_1\}$ generate an ultrafilter on ω_1.*

15. REMARK. Laver informed us kindly in [16] that Woodin had showed the relative consistency of ($2^{\omega}=\omega_1$ + there is a σ-complete nontrivial ideal i on ω_1 such that $\mathscr{P}(\omega_1)/i \cong Col(\omega, \omega_1)$). Hence the negation of any of the statements in Prop. 13 is relatively consistent.

There is yet another particular solution of Fodor's problem, namely

16. PROPOSITION. *If there is no ω_1-scale in ${}^{\omega}\omega$ and if $\mathscr{A}=\{A_\alpha : \alpha<\omega_1\}$ is a family of stationary subsets of ω_1 closed under finite intersections, then $\mathscr{A}$ has a disjoint refinement consisting of stationary sets.*

17. REMARK. Laver making use of Kunen's model [14] showed the relative consistency of (There is no ω_1-scale in ${}^{\omega}\omega$ + there is a σ-complete non-trivial ideal i on ω_1 such that sat $(\mathscr{P}(\omega_1)/i)=\omega_2$.) Hence, in this model no σ-complete non-trivial ideal on $\mathscr{P}(\omega_1)$ can be extended into a maximal one by adding ω_1 generators only.

Finally, let us give a solution of a problem posed by Woods [22], Hušek [12] and van Douwen (see [8]).

18. THEOREM. *Let κ, λ be cardinals, λ regular, let $\omega_1 \leqslant \lambda \leqslant 2^{\kappa}$. Then there exist a filter $\mathscr{F}$ and an ultrafilter $\mathscr{U}$ on $\mathscr{P}(\kappa)/i_F$ such that the smallest possible cardinality of a family $\mathscr{A}\subset\mathscr{P}(\kappa)/i_F$ such that $\mathscr{F}\cup\mathscr{A}$ generates $\mathscr{U}$ is $|\mathscr{A}|=\lambda$.*

19. REMARK. Similar result was proved by Shelah and Kunen, independently, for $\lambda=\omega_1$ [17, 13].

20. PROBLEMS.

(a) Let $\mathscr{U}$ be an ultrafilter on $\mathscr{P}(\kappa)/i_F$, where κ is an infinite cardinal. By a $\mathscr{U}$-tower we understand a sequence $\mathscr{T}=\{t_\alpha : \alpha\in\lambda\}\subset\mathscr{U}$ (the cardinal λ is called the length of $\mathscr{T}$) such that $t_\alpha \geqslant t_\beta$ whenever $\alpha<\beta<\lambda$ and for each $v\in\mathscr{U}$ there is some $\alpha<\lambda$ with $v-t_\alpha\neq\mathbf{0}$. (Compare with the slightly different notion of a tower—that is a sequence $\mathscr{T}=\{t_\alpha : \alpha\in\lambda\}\subset\mathscr{P}(\kappa)/i_F$ such that $t_\alpha\geqslant t_\beta$ whenever $\alpha<\beta<\lambda$ and for each $x\in\mathscr{P}(\kappa)/i_F$, $x\neq\mathbf{0}$ there is an $\alpha<\lambda$ with $x-t_\alpha\neq\mathbf{0}$.). It is known that under (CH) there is an ultrafilter $\mathscr{U}$ on $\mathscr{P}(\omega)/i_F$ containing no tower of an uncountable length, [15]. Despite this, each ultrafilter $\mathscr{U}$ on $\mathscr{P}(\omega)/i_F$ contains some $\mathscr{U}$-tower $\mathscr{T}$ whose length is of cofinality greater than ω, this is true in ZFC. Does each ultrafilter $\mathscr{U}$ on $\mathscr{P}(\kappa)/i_F$ for uncountable regular κ contain a $\mathscr{U}$-tower $\mathscr{T}$ with cf $(|\mathscr{T}|)>\kappa$?

(b) Let κ be uncountable regular cardinal. Define a family $\{P_\alpha;\ \alpha\in\kappa^+\}\subset\mathscr{P}(\kappa)$ to be (κ^+, ω, i_F)-regular if for each infinite $x\subset\kappa^+$, $|\bigcap\{P_\alpha : \alpha\in x\}|<\kappa$. There always exists a uniform ultrafilter $\mathscr{U}$ on κ, which contains a (κ^+, ω, i_F)-regular family. (Prikry observed that no such ultrafilter exists on ω). Is it consistent that each ultrafilter on κ uncountable regular contains a (κ^+, ω, i_F)-regular family ?

(c) Consider the Boolean algebra $\mathscr{P}(\kappa)/i_F$ where κ is uncountable. What is the smallest cardinality of a dense subset of $\mathscr{P}(\kappa)/i_F$? Can it be $<2_\kappa$? (There is a model in [1], where $2^{\omega_1}=|P(\omega_1)/i_F|=\omega_3$, sat $(\mathscr{P}(\omega_1)/i_F)=\omega_3$. But there is no dense subset of $\mathscr{P}(\omega_1)/i_F$ of cardinality $<2^{\omega_1}$).

(d) Let κ be an uncountable cardinal. Can sat $(\mathscr{P}(\kappa)/i_F)$ be a weakly inaccessible cardinal?

(e) By a base matrix for a Boolean algebra $\mathscr{B}$ we understand a collection Θ of partitions of $\mathscr{B}$ such that for each non-zero $a \in \mathscr{B}$ there is some $p \in \Theta$ and some $x \in p$ with $0 \neq x \leqslant a$. Let μ be the smallest cardinality of a base matrix for $\mathscr{B}$, d the smallest cardinality of a dense subset of $\mathscr{B}$ and $\sigma = \text{sat}(\mathscr{B})$. What are the non-trivial possible relations between μ, d and σ for Boolean algebras homogeneous in μ, d and σ, especially for choice $\mathscr{B} = \mathscr{P}(\kappa)/i_F$, κ uncountable? (Under (GCH), we know: $\sigma = (2^\kappa)^+$, $d = 2^\kappa$, $\mu = \omega$ if $\text{cf}(\kappa) \neq \omega$, $\mu = \omega_1$ if $\text{cf}(\kappa) = \omega$. But in the model given in [1] is $\mu = 2^{\omega_1}$ for $\mathscr{P}(\omega_1)/i_F$. The case $\mathscr{P}(\omega)/i_F$ was discussed in [5].)

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REFERENCES

[1] J. E. Baumgartner, *Almost disjoint sets, the dense set problem and the partition calculus*, Ann. Math. Logic, **10** (1976), 401–439.

[2] B. Balcar, F. Franek, *Independent families on complete Boolean algebras*, [to appear].

[3] B. Balcar, R. Frankiewicz, *Ultrafilters and ω_1-points in βN-N*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys., **27** (1979), 593–598.

[4] J. E. Baumgartner, A. Hajnal, A. Máté, *Weak saturation properties of ideals*, in: *Infinite and finite sets*, vol. I, ed. by A. Hajnal, NHPC, Amsterdam, 137–158.

[5] B. Balcar, J. Pelant, P. Simon, *The space of ultrafilters on N covered by nowhere dense sets*, Fund. Math., **110** (1980), 11–24.

[6] B. Balcar, P. Vojtáš, *Refining systems on Boolean algebras*, in: *Set theory and hierarchy theory*, Lect. Notes in Math, **619**, Springer, Berlin, 1977, 45–58.

[7] — , — , *Almost disjoint refinement of families of subsets of N*, Proc. Amer. Math. Soc., **79**, 3 (1980), 465–470.

[8] W. W. Comfort, *Compactifications. Recent results from several countries*, in: *Topology Proceedings*, vol. 2, 1977, 61–87.

[9] W. W. Comfort, N. B. Hindman, *Refining families for ultrafilters*, Math. Z., **149** (1976), 189–199.

[10] W. W. Comfort, S. Negrepontis, *The theory of ultrafilters*, Springer, Berlin, 1974.

[11] N. B. Hindman, *On the existence of c-point in βN-N*, Proc. Amer. Math. Soc., **21** (1969), 277–280.

[12] M. Hušek, *Topological spaces without κ-accessible diagonal*, Comment. Math. Univ. Carolinae, **18**, 4 (1977), 777–787.

[13] K. Kunen, Letter to the author, Sept. 14, 1978.

[14] — , *Saturated ideals*, J. Symbolic Logic, **43** (1), (1978), 65–76.

[15] K. Kunen, J. van Mill, Ch. F. Mills, *On nowhere dense closed P-sets*, Proc. Amer. Math. Soc., **78** (1980, 119–123.

[16] R. Laver, Letter to the author, Dec. 1978.

[17] S. Shelah, *Some consistency results in topology* [manuscript].

[18] W. Sierpiński, *Hypothèse du continu*, 2nd ed., New York, 1956.

[19] K. Prikry, *Ultrafilters and almost disjoint sets II*, Bull. Amer. Math. Soc., 81 (1975), 209–212.

[20] — , *On refinements of ultrafilters* [typescript].

[21] A. D. Taylor, *Regularity properties of ideals and ultrafilters*, Ann. Math. Logic, 16 (1979), 33–35.

[22] R. G. Woods, *Characterization of some C^*-embedded subspace of βN*, Pacific J. Math., **65** (1976), 573–579.

Б. Балцар, П. Симон, П. Войташ, **Свойства подразделения и разширения фильтров**

Содержание. Мы сообщаем несколько результатов касающихся свойства непересекающихся подразделений в булевых алгебрах, подчеркивая системы подмножеств данного множества по модулю какого-нибудь идеала. Мы ответили на вопросы Комфорта и Гиндмана, Вудса и Хушка и Фан Дауэна. Эту работу приводим без доказательств.