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ALMOST DISJOINT REFINEMENT OF FAMILIES OF SUBSETS OF N

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ABSTRACT. Without any set-theoretic assumptions, we prove that every uniform ultrafilter on the set N of all natural numbers has a Comfort system, that is, an almost disjoint refinement. Moreover, we describe one type of ideal such that the family of all subsets of N that are not contained in it has an almost disjoint refinement.

1. The problem and the theorems. For a cardinal number $\nu > 1$, Hechler has generalized Pierce's notion of a ν-point to a ν-set of a topological space. A nonempty subset S of a topological space X is called a ν-set if there exists a family of ν pairwise disjoint open sets, each of which contains S in its closure. A point $p \in X$ is a ν-point of X if the singleton $\{p\}$ is a ν-set. We concentrate on the space $\beta N - N = N^*$ of uniform ultrafilters on the set N of all natural numbers.

The problem whether each point of N^* is a 2^ω-point or, more generally, whether each nowhere dense subset of N^* is a 2^ω-set, has a little longer history (cf. Pierce [P], Hindman [H], Comfort [CH], van Douwen [vD], Roitman [R], Kunen [K], Szymanski [Sz], Hechler [H], Frankiewicz [BF] and others). Short historical remarks can be found in [CH] or [BF].

1.1 Without any additional set-theoretic assumptions, we shall prove that every point of N^* is a 2^ω-point. This gives an affirmative answer to a problem raised by Comfort and Hindman [CH]. We also describe a type of nowhere dense subsets of N^* that are 2^ω-sets. The main problem of Hechler's paper [H], whether every nowhere dense subset of N^* is a 2^ω-set, remains open.

1.2. For a set A let $[A]^\omega$ be the set of all denumerable subsets of A; the notation $A \subseteq^* B$ means that $A - B$ is finite. We say that a family $\{A_\alpha : \alpha < \nu\}$ of subsets of a set X is a tower on X of length ν if $A_\alpha \subseteq^* A_\beta$ for $\alpha > \beta$. We say that a set C is a selector of a family $\{q_n : n \in \omega\}$ if C is infinite, $C \subseteq \bigcup \{q_n : n \in \omega\}$, and $|C \cap q_n| \leqslant 1$ for $n \in \omega$. $\mathcal{I}_F$ denotes the ideal of all finite subsets of N. In this paper all ideals are assumed to be proper and to contain $\mathcal{I}_F$. Sets from $\mathcal{I}^+ = \mathcal{P}(N) - \mathcal{I}$ are called large sets with respect to $\mathcal{I}$ for an ideal $\mathcal{I}$.

1.3. We shall deal with families $\mathcal{Q} \subseteq [N]^\omega$ and we look for $\mathcal{Q}$ that have an almost disjoint refinement (ADR) i.e. a family $\{C_X : X \in \mathcal{Q}\}$ such that

(i) $C_X \in [X]^\omega$,

(ii) for $X \neq Y$ the set $C_X \cap C_Y$ is finite.

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Recall that $\mathcal{P} \subseteq [N]^\omega$ is a MAD family on N if $\mathcal{P}$ is an infinite maximal family of pairwise almost disjoint infinite subsets of N.

The following facts are well known.

(i) A family $\mathcal{A} \subseteq [N]^\omega$ has an ADR iff there is a MADF $\mathcal{P}$ such that for every $A \in \mathcal{A}$ we have

$$|\{X \in \mathcal{P} : X \cap A \text{ is infinite}\}| = 2^\omega.$$

(ii) Let $\mathcal{U}$ be a uniform ultrafilter on N. Then $\mathcal{U}$ as a point of N^* is a 2^ω-point of N^* iff there is an ADR for $\mathcal{U}$.

1.4. Following Mathias [M] we shall say that an ideal $\mathcal{I}$ on N is tall if for all $X \in [N]^\omega$ there is a $Y \in [X]^\omega$ with $Y \in \mathcal{I}$. Let $\mathcal{P}$ be a MAD family; then $\mathcal{I}(\mathcal{P})$ denotes the ideal generated by $\mathcal{I}_F \cup \mathcal{P}$.

DEFINITION. Let $Q = \{q_n : n \in \omega\}$ be a partition on N into infinitely many (finite or infinite) pieces such that for all $k \in \omega$ there are infinitely many q_n with at least k elements. Let $\mathcal{Y}(Q)$ be the ideal generated by the union of the sets $\{X \subseteq N: (\exists k)(\forall n \in \omega)(|X \cap q_n| \leqslant k)\}$ and $\{q_n : n \in \Omega\}$. In [M] it is shown that the ideals $\mathcal{I}(\mathcal{P})$ and $\mathcal{Y}(Q)$ are both tall. It is easily seen that:

(a) If $\mathcal{A}$ is a family with an ADR, then for every $X \in [N]^\omega$ there is $Y \in [X]^\omega$ such that $[Y]^\omega \cap \mathcal{A} = \emptyset$.

(b) If $\mathcal{A}$ is a family with an ADR then there is a MAD family $\mathcal{P}$ such that $\mathcal{A} \subseteq \mathcal{I}^+(\mathcal{P})$.

(c) Tall ideals correspond to open dense subsets of N^* that are not the whole space. If $\mathcal{I}$ is a tall ideal then $\mathcal{I}^+$ has an ADR iff the complement of the open set corresponding to $\mathcal{I}$ is a 2^ω-set.

(d) The extremal problem whether for every MADF $\mathcal{P}$ there exists an ADR for $\mathcal{I}^+(\mathcal{P})$ is equivalent to the above-mentioned problem of Hechler.

1.5. THEOREM A. *Let Q be a partition of N as in Definition* 1.4. *Then the family $\mathcal{Y}^+(Q)$ has an ADR.*

As a straightforward corollary we obtain that every nonselective uniform ultrafilter on N has an ADR. For ultrafilters however we shall prove a little more.

1.6. THEOREM B. *Let $\mathcal{F}$ be a uniform ultrafilter on N and $\mathcal{P}$ a MAD family of N with $\mathcal{P} \cap \mathcal{F} = \emptyset$. Then there is an ADR for $\mathcal{F}$ which consists of large sets with respect to the ideal $\mathcal{I}(\mathcal{P})$.*

1.7. COROLLARY. *Since the ideal $\mathcal{F}^*$ dual to a uniform ultrafilter $\mathcal{F}$ is tall, there is a MADF $\mathcal{P} \subseteq \mathcal{F}^*$. Thus every uniform ultrafilter $\mathcal{F}$ has an ADR.*

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2. Proofs of the theorems. We begin with some cardinal characteristics.

2.1. For functions from N to N consider the preordering $f <^* g$ iff $\{n: f(n) \geqslant g(n)\}$ is finite. The least cardinal of a family of functions that is unbounded under $<^*$ is denoted by λ. Obviously there is a family of functions $\{f_\alpha : \alpha \in \lambda\}$

unbounded under $<^*$ such that f_α's are increasing and $\alpha < \beta$ implies $f_\alpha <^* f_\beta$. Note that every set of functions with cardinality less than λ has an $<^*$-upper bound. Due to this fact, for every partition $\{X_n: n \in \omega\}$ of N whose members are infinite, there is a tower $\{A_\alpha: \alpha \in \lambda\}$ such that

(i) $X_n - A_\alpha$ is finite for every n, α;

(ii) for any $X \in [N]^\omega$, if $\{n: |X_n \cap X| = \aleph_0\}$ is infinite, then $(\exists \alpha \in \lambda)(X \not\subseteq^* A_\alpha)$.

2.2. The following is defined in [**BPS**]. Let κ denote the least cardinal such that the Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}_F$ of all subsets of ω modulo finite sets is not (κ, ∞)-distributive. The "Base matrix theorem" proved in [**BPS**] says: There is a system $\{\mathcal{P}_\alpha: \alpha \in \kappa\}$ of MAD families such that for $\alpha > \beta$, $\mathcal{P}_\alpha$ *-refines $\mathcal{P}_\beta$ and for every $A \in [\omega]^\omega$ there is a $B \in \bigcup \{\mathcal{P}_\alpha: \alpha \in \kappa\}$ such that $B \subseteq A$.

2.3. Let d denote the minimal cardinal such that there is a MADF $\mathcal{P}$ on N with $|\mathcal{P}| = d$.

Lemma. $\omega_1 \leqslant \kappa \leqslant d$.

Proof. In [**BPS**] the inequalities $\omega_1 \leqslant \kappa \leqslant \lambda$ are proved. The proof is finished by adding the known inequality $\lambda \leqslant d$; see [**So**].

Remember that under the assumption $d = 2^\omega$ Hechler's conjecture is known to be true [**R**], [**H**].

2.4. The following lemma plays a key role in our proofs.

Lemma. *There is a family* $\mathcal{B} \subseteq [\omega]^\omega$ *such that the following conditions hold for any* $u, v \in \mathcal{B}$.

(i) $u \cap v =^* \emptyset$ *or* $u \subseteq^* v$ *or* $v \subseteq^* u$;

(ii) $|\{w \in \mathcal{B} : u \subseteq^* w\}| < \kappa$;

(iii) *for any* $X \in [\omega]^\omega$ *there is a* $w \in B$ *such that* $w \subseteq X$.

Proof. Let $\{\mathcal{P}_\alpha: \alpha < \kappa\}$ be the base matrix mentioned in 2.2. Then $\mathcal{B} = \bigcup \{\mathcal{P}_\alpha: \alpha < \kappa\}$ has the desired properties.

2.5. Lemma. *Assume* $\mathcal{B}$ *is as in Lemma* 2.4, $\mathcal{B}_0 \subseteq \mathcal{B}$ *such that* $|\mathcal{B}_0| < 2^\omega$. *Then for every* $C \in [\omega]^\omega$ *there is a* $u \in \mathcal{B} - \mathcal{B}_0$ *such that* $u \subseteq C$ *and* $(\forall v \in \mathcal{B}_0)(v \cap u =^* \emptyset$ *or* $u \subseteq^* v)$.

Proof. As there is a MAD family on ω of cardinality 2^ω, by (iii) of Lemma 2.4 we have $|\{v \in \mathcal{B}: v \subseteq^* u\}| = 2^\omega$ for every $u \in \mathcal{B}$. This fact with (i) of 2.4 finishes the proof.

2.6. Proof of Theorem A. Let $Q = \{q_n: n \in \omega\}$ be a partition of N as in Definition 1.4. Put $\{A_\alpha: \alpha < 2^\omega\} = \mathcal{Y}^+(Q)$. For A_α we shall now pick a set $c(\alpha)$ as follows. If $X = \{i \in \omega: |q_i \cap A_\alpha| = \aleph_0\}$ is infinite then we put $c(\alpha) = X$. Otherwise we pick $c(\alpha) \in [\omega - X]^\omega$ such that $i < j$ implies $|A_\alpha \cap q_i| < |A_\alpha \cap q_j|$ for $i, j \in c(\alpha)$. Let $\mathcal{B}$ be the Base family from Lemma 2.4. In the sequel $\mathcal{B}$ is used on ω as indexes of $\{q_i: i \in \omega\}$. By transfinite recursion through $\alpha < 2^\omega$ we shall define $F(\alpha) \in [A_\alpha]^\omega$ and $I(\alpha) \subseteq c(\alpha)$ such that

(i) $I(\alpha) \in \mathcal{B}$;

(ii) $F(\alpha)$ is a selector for $R_\alpha = \{q_i \cap A_\alpha: i \in I(\alpha)\}$ i.e. $F(\alpha) \subseteq \bigcup R_\alpha$ and $|F(\alpha) \cap q_i \cap A_\alpha| \leqslant 1$ for any $i \in I(\alpha)$;

(iii) $F(\alpha)$ is almost disjoint with all $F(\beta)$, and $I(\alpha) \neq^* I(\beta)$ for $\beta < \alpha$.

For $\alpha < 2^\omega$ we put $D_\alpha = \bigcup R_\alpha$.

Step 0. There is $I(0) \in \mathfrak{B}$ such that $I(0) \subseteq c(0)$. As $F(0)$ take an (infinite) selector of R_0.

Step $\alpha < 2^\omega$. For $\beta, \gamma < \alpha$ we have $F(\beta), I(\beta)$ such that $I(\beta) \neq^* I(\gamma)$ and $F(\beta) \cap F(\gamma)$ is finite for $\beta \neq \gamma$. Choose $I(\alpha)$ using 2.5 with respect to $\mathfrak{B}_0 = \{I(\beta): \beta < \alpha\}$ and *-different from all $I(\beta)$. Put $\mathcal{V} = \{F(\beta) \cap D_\alpha: \beta < \alpha$ and $I(\alpha) \subseteq^* I(\beta)$ and $|F(\beta) \cap D_\alpha| = \aleph_0\} \cup (R_\alpha \cap [N]^\omega)$.

Members of $\mathcal{V}$ are pairwise almost disjoint. According to the choice of $c(\alpha)$ and since $F(\beta)$'s are selectors, D_α cannot be $=^*$ to the union of any finite part of $\mathcal{V}$. By (ii) of Lemma 2.4 we have $|\mathcal{V}| < \kappa \leqslant d$. Hence there is $F(\alpha) \subseteq D_\alpha$, an infinite selector of R_α that is almost disjoint with all members of $\mathcal{V}$. We note that $I(\alpha) \cap I(\beta) =^* \varnothing$ implies $F(\alpha) \cap F(\beta) =^* \varnothing$. Hence $\{F(\alpha): \alpha < 2^\omega\}$ is an ADR for $\mathcal{Y}^+(Q)$. The proof of Theorem A is complete.

2.7. Our starting point for the proof of Theorem B is the notion of an ultrafilter's tower.

Definition. A tower $\mathfrak{A} = \{A(\alpha): \alpha \in \nu\}$ is a tower of a uniform ultrafilter $\mathfrak{F}$ if

(i) ν is uncountable and regular;

(ii) $\mathfrak{A} \subseteq \mathfrak{F}$;

(iii) for any $X \in \mathfrak{F}$ there is $\alpha \in \nu$ such that $X \not\subseteq^* A(\alpha)$.

2.8. Lemma. *Assume $\mathfrak{F}$ is a uniform ultrafilter on N. Then there is a tower of $\mathfrak{F}$ (of uncountable length).*

Proof. It is clear that such a tower exists for P-ultrafilters. In the case of non-P-ultrafilters we can take a tower of the length λ from 2.1, where the partition is the one exemplifying the non-P-property.

2.9. Lemma. *Assume $\{A(\alpha): \alpha \in \nu\}$ is a tower of $\mathfrak{F}$ and $\mathfrak{P}$ is a MAD family such that $\mathfrak{F} \cap \mathfrak{P} = \varnothing$. Then*

(iv) $(\forall \alpha \in \nu)(\exists \beta > \alpha)(A(\alpha) - A(\beta) \in \mathfrak{J}^+(\mathfrak{P}))$.

Proof. By induction we can choose an increasing sequence $\{\alpha_n: n \in \omega\}$ and a family $\{u_n: n \in \omega\}$ of different elements of $\mathfrak{P}$ such that $(A(\alpha_i) - A(\alpha_{i+1})) \cap u_i$ is infinite. For $n \in \omega$ we have $A(\alpha_n) - \bigcup \{u_i: 0 \leqslant i \leqslant n-1\} \in \mathfrak{F}$. Hence by (iii) of Definition 2.7 there are $\alpha_{n+1} > \alpha_n$ and $u_n \in \mathfrak{P} - \{u_0, \ldots, u_{n-1}\}$ such that $(A(\alpha_n) - A(\alpha_{n+1})) \cap u_n$ is infinite. Put $\beta = \sup\{\alpha_n: n \in \omega\}$.

2.10. Proof of Theorem B. Let $\mathfrak{F}$ be a uniform ultrafilter and $\mathfrak{P}$ a MADF with $\mathfrak{P} \cap \mathfrak{F} = \varnothing$. Assume $\{A(\alpha): \alpha < \nu\}$ is a tower of $\mathfrak{F}$ with $A(\alpha) - A(\beta) \in \mathfrak{J}^+(\mathfrak{P})$ for $\alpha < \beta$. We put $\nu(\omega) = \{\alpha \in \nu: \operatorname{cf}(\alpha) = \omega\}$. For every $\alpha \in \nu(\omega)$ we fix an increasing sequence $\{\alpha_n: n \in \omega\}$ such that $\alpha = \sup\{\alpha_n: n \in \omega\}$. We define $q(\alpha, n) = \bigcap \{A(\alpha_i): 0 \leqslant i \leqslant n\} - (A(\alpha_{n+1}) \cup A(\alpha))$. Note that $q(\alpha, n) \in \mathfrak{J}^+(\mathfrak{P})$ and $q(\alpha, n) \cap q(\alpha, m) = \varnothing$ for all $\alpha, n \neq m$. It is easy to see by Lemma 2.9 that for every $X \in \mathfrak{F}$ there is $\alpha \in \nu(\omega)$ such that the set $\{n: X \cap q(\alpha, n) \in \mathfrak{J}^+(\mathfrak{P})\}$ is infinite.

We define $Q_\alpha = \{q(\alpha, n)\colon n \in \omega\}$ for $\alpha \in \nu(\omega)$. If K_α, K_β are selectors of Q_α, Q_β respectively and $\alpha \neq \beta$ then $K_\alpha \cap K_\beta$ is finite. Hence for the proof of Theorem B it suffices to show that the family $\mathcal{S}(Q_\alpha) = \{X \subseteq N\colon |\{n\colon q(\alpha, n) \cap X \in \mathcal{I}^+(\mathcal{P})\}| = \aleph_0\}$ has an ADR consisting of selectors of Q_α which are large sets. The argument is now similar to the one used in 2.6. Let $Q = \{q_n\colon n \in \omega\} = Q_\alpha$ for $\alpha \in \nu(\omega)$ and let $\{D(\alpha)\colon \alpha < 2^\omega\}$ be a numbering of $\mathcal{S}(Q)$. We put $c(\alpha) = \{i\colon D(\alpha) \cap q_i \in \mathcal{I}^+(\mathcal{P})\}$. By transfinite recursion we define sets $I(\alpha)$, $F(\alpha)$ such that

(i) $I(\alpha) \in \mathcal{B}$, where $\mathcal{B}$ is the Base family from Lemma 2.4 and $I(\alpha) \subseteq c(\alpha)$;

(ii) $F(\alpha)$ is a selector for $\{q_i \cap D_\alpha\colon i \in I(\alpha)\}$ and $F(\alpha) \in \mathcal{I}^+(\mathcal{P})$;

(iii) for $\beta < \alpha$, $F(\alpha) \cap F(\beta)$ is finite and $I(\beta) \cap I(\alpha) =^* \varnothing$ or ($I(\alpha) \subseteq^* I(\beta)$ and $I(\alpha) \neq^* I(\beta)$).

In the step $\alpha < 2^\omega$ we choose $I(\alpha) \in \mathcal{B}$ using Lemma 2.5. The $F(\beta)$'s are selectors and hence they determine partial functions f_β on ω. We set $\mathcal{V} = \{f_\beta \cap (I(\alpha) \times D(\alpha))\colon \beta < \alpha, \text{and } I(\alpha) \subseteq^* I(\beta)\}$. As $|\mathcal{V}| < \kappa \leqslant \lambda$ there is a function $f\colon I(\alpha) \to N$ that is an $\leqslant^*$-upper bound for $\mathcal{V}$. We note that for any infinite family of pairwise disjoint large sets there is a large selector. Hence we may take $F(\alpha)$ as a large selector of the family $\{D_\alpha \cap q_i - \{n\colon n \leqslant f(i)\}\colon i \in I(\alpha)\}$. It is obvious that $F(\alpha) \cap F(\beta)$ is finite for $\beta < \alpha$.

This completes the proof.

3. Remarks and problems.

3.1. We do not know if the following observation is known. Let us consider a MAD family on the set Q of all rational numbers in the unit interval [0, 1] of the real line. Then there is a MAD family $\mathcal{P}$ on Q such that for any set $A \subseteq Q$ that has infinitely many accumulation points in the space [0, 1] there exists $B \in \mathcal{P}$ with $B \subseteq A$. This fact follows from Theorem A.

3.2. Let $s = \{a_n\colon n \in N\}$ be a sequence of positive reals with $\lim a_n = 0$ and $\Sigma a_n = \infty$. Let us consider the ideal $\mathcal{Y}(s) = \{X \subseteq N\colon \Sigma\{a_n\colon n \in X\} < \infty\}$. This type of ideal seems to be similar to the ideal of type $\mathcal{Y}(Q)$ from Definition 1.4, where Q is partition of N consisting of finite sets. But we do not know whether $\mathcal{Y}^+(s)$ has an ADR.

3.3. Does the assumption "every $<^*$-cofinal subset of functions from N to N has cardinality 2^ω" imply Hechler's conjecture?

3.4. Consider the Boolean algebra $B = \mathcal{P}(N)/\mathcal{I}_F$. Corollary 1.7 is equivalent to the statement "every filter base on B of cardinality at most 2^ω has a disjoint refinement". This statement cannot be strenghtened to the completion $\overline{B}$ of the algebra B. Using a result of Kunen, van Mill and Mills [**KvMM**] in [**BSV**] have proved the following.

If $2^\tau \leqslant 2^\omega$ for all $\tau < 2^\omega$ then there is an ultrafilter on $\overline{B}$ with a base of cardinality 2^ω. Then there is no disjoint refinement on $\overline{B}$ for any base of this ultrafilter.

3.5. K. Kunen, using an observation from [**BF**], proved the following generalization of a result in [**BF**]. If X is any compact space in which nonempty G_δ sets have nonempty interior, then very nonisolated point in X is an ω_1-point. He also has remarked that for the above class of spaces we cannot replace ω_1 by 2^ω.

Let us consider only spaces that moreover have no isolated point. Are there any simple conditions on such spaces that imply "every point is a 2^ω-point"? We note that for arbitrary $\tau \geqslant \omega$, every point of the space $\beta(\tau) - \tau$ is a 2^ω-point.

References

[BF] B. Balcar and R. Frankiewicz, *Ultrafilters and ω_1-points in $\beta N - N$* (to appear).

[BSV] B. Balcar, P. Simon and P. Vojtáš, *Refinement properties and extending of filters in Boolean algebras* (to appear).

[BPS] B. Balcar, J. Pelant and P. Simon, *The space of ultrafilters on N covered by nowhere dense sets*, Fund. Math. (to appear).

[CH] W. W. Comfort and N. B. Hindman, *Refining families for ultrafilters*, Math. Z. **149** (1976), 189–199.

[vD] E. K. van Douwen, *Martin's axiom and pathological points in $\beta X - X$* (manuscript).

[H] S. H. Hechler, *Generalization of almost disjointness, c-sets, and the Baire number of $\beta N - N$*, General Topology and Appl. **8** (1978), 93–110.

[Hd] N. B. Hindman, *On the existence of $\mathfrak{c}$-point in $\beta N - N$*, Proc. Amer. Math. Soc. **21** (1969), 277–280.

[J] T. J. Jech, *Set theory*, Academic Press, New York, 1978.

[K] K. Kunen, Letter to the author, Sept. 14, 1978.

[KvMM] K. Kunen, J. van Mill and Ch. F. Mills, *On nowhere dense closed P-sets* (to appear).

[M] A. R. D. Mathias, *Happy families*, Ann. Math. Logic **12** (1977), 59–111.

[P] R. C. Pierce, *Modules over commutative regular rings*, Mem. Amer. Math. Soc. no. 70, 1967.

[R] J. Roitman, *Almost disjoint strong refinements*, Notices Amer. Math. Soc. **22** (1975), A 328.

[So] R. C. Solomon, *Families of sets and functions*, Czechoslovak Math. J. **27 (102)** (1977), 556–559.

[Sz] A. Szymanski, *On the existence of $\aleph_0$-points*, Proc. Amer. Math. Soc. **66** (1977), 128–130.

[T] A. D. Taylor, *Regularity properties of ideals and ultrafilters*, Ann. Math. Logic **16** (1979), 33–55.

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