REFINING SYSTEMS ON BOOLEAN ALGEBRAS

by

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Introduction. The present paper contains some observations and problems concerning disjoint systems refining a given system. One of the first classical results in this area is the following theorem of Sierpiński: If κ is an infinite cardinal and $u = \{u_\alpha ; \alpha \in \kappa\}$ is a system of sets of cardinality κ then there is a disjoint refining system $\{v_\alpha ; \alpha \in \kappa\}$ for u, this means $|v_\alpha| = \kappa$, $v_\alpha \subseteq u_\alpha$ and $\alpha \neq \beta$ implies $v_\alpha \cap v_\beta = 0$. The problem of disjoint refinements is developed in [5],[2],[1],[7],[8],[9] and [11]. In section 1 we define the disjoint refining property $Rf(\underline{b},\kappa)$ of a Boolean algebra $\underline{b}$ with parameter κ, and $Rfip(\underline{b},\kappa)$, the disjoint refining property of $\underline{b}$ with respect to systems satysfying the finite intersection property. We describe a class of algebras where $Rf(\underline{b},\kappa)$ is equivalent to the extremal condition on the cardinality of a basis of $\underline{b}$ (Theorem 1.6). If $\kappa^{<\kappa} = \kappa$ then for Boolean algebras with a basis of cardinality at most κ we show a connection between $Rfip(\underline{b},\kappa)$ and the minimal cardinality of a basis of ultrafilters on $\underline{b}$ (Theorem 1.11). In section 2 the above mentioned properties are applied to problems concerning σ-complete ideals on ω_1.
We can not settle the following problem:

Does there exist a complete Boolean algebra with a basis of cardinality $\kappa > \aleph_0$, which has the $Rfip(\kappa)$ property?

0. Preliminaries.

We use the usual system of Set theory with the axiom of choice. Infinite cardinals are denoted by κ, λ. If $u : x \rightarrow y$ is a mapping from x to y, then we often write u_v instead of $u(v)$ for $v \in x$. We assume fundamental facts from the theory of Boolean algebras, [13],[3]. The set of all nonzero elements of a Boolean algebra $\underline{b}$ is denoted b^+. By the canonical ordering of $\underline{b}$ we mean the relation $x \leq_b y$ iff $x \wedge_b y = x$. The fact that algebras $\underline{b}_1, \underline{b}_2$ are isomorphic is expressed by $\underline{b}_1 \cong \underline{b}_2$. By a partial Boolean algebra $\underline{b}_x$ of $\underline{b}$ for $x \in b^+$ we understand an algebra with the universe

$b_x = \{y \in b; y \leq_b x\}$ and restricted operations. p is a partition of $\underline{b}$ if $p \subseteq b^+$ and elements of p are pairwise disjoint. p is a maximal partition if, in addition, $\bigvee p = \mathbb{1}_b$. For $x \in b^+$ and any partitions $p, q \subseteq b^+$ we put $p \wedge\wedge x = \{x \bar{\wedge} v; x \wedge v \neq \mathbb{O}_b \;\&\; v \in p\}$. Analogously $p \wedge\wedge q = \{y;\ y \neq \mathbb{O} \;\&\; (\exists v \in p)(\exists z \in q)(\ y = v \bar{\wedge} z)\}$. A system $u \subseteq b^+$ has the finite intersection property (Fip(u)) if for every finite, nonempty, $v \subseteq u$ we have $\bigwedge v \neq \mathbb{O}_b$. Saying that i is an ideal on $\underline{b}$ means that i is a proper ideal, i.e. $\mathbb{1}_b \notin i$, and similarly for filters. For each $x \in b^+$, sat(x) is the least cardinal λ such that there is no partition of $\underline{b}_x$ of cardinality λ. In contrast to the traditional notion of saturation we define the saturation of a Boolean algebra $\underline{b}$ as $\mathrm{sat}(\underline{b}) = \min\{\mathrm{sat}(x);\ x \in b^+\}$. $u \subseteq b^+$ is a basis of $\underline{b}$ if for every $x \in b^+$ there is $y \in u$ such that $y \leq x$. Assume j is a filter on $\underline{b}$. Then $u \subseteq j$ is a basis of j if for every $x \in j$ there is $y \in u$ such that $y \leq x$. For each Boolean algebra $\underline{b}$, Comp($\underline{b}$) denotes the complete Boolean algebra with base b. Note that Comp($\underline{b}$) is determined uniquely up to isomorphism and that $\underline{b}$ is a subalgebra of Comp($\underline{b}$). Consider $\varkappa$ with the discrete topology and the product P of ω copies of $\varkappa$ with the product topology. Then the system of all regular open sets in P forms a complete Boolean algebra, denoted by $\mathrm{Col}(\omega,\varkappa)$. Put $d = \{f;\ f: n \to \varkappa \;\&\; n \in \omega\}$ with the partial ordering $f \leq g$ iff $f \supseteq g$. Then d is isomorphic to a basis $\mathrm{Col}(\omega,\varkappa)$ with respect to $\leq$ and the canonical ordering of $\mathrm{Col}(\omega,\varkappa)$. Remember, if two complete Boolean algebras have isomorphic bases then they are isomorphic.

1. Refinements for families.

1.1. DEFINITION. Let $\varkappa$ be a cardinal, $\underline{b}$ a Boolean algebra.

(i) A mapping $u: a \to b^+$ has a disjoint refining system, (in symbols $\mathrm{Rf}(\underline{b},u)$ or briefly Rf(u)), if there exists a $v: a \to b^+$ such that $v(x) \leq u(x)$ and $x \neq y$ implies $v(x) \wedge v(y) = \mathbb{O}$. We often say refining system instead of disjoint refining system.

(ii) $\underline{b}$ has the disjoint refinement property for systems of cardinality $\varkappa$ (in symbols $\mathrm{Rf}(\underline{b},\varkappa)$), if Rf(u) holds for all $u: \varkappa \to b^+$.

(iii) $\underline{b}$ has the disjoint refinement property for systems of cardinality $\varkappa$ satisfying Fip (in symbols $\mathrm{Rfip}(\underline{b},\varkappa)$), iff for every $u: \varkappa \to b^+$ such that $\mathrm{Fip}(\{u_\alpha;\ \alpha \in \varkappa\})$ we have Rf(u).

1.2. REMARK. (i) $Rf(\underline{b},\varkappa) \rightarrow Rfip(\underline{b},\varkappa)$ and there are algebras such that Rf is stronger than Rfip.

(ii) $Rfip(\underline{b},\varkappa) \rightarrow sat(\underline{b}) > \varkappa$.

(iii) Let $sat(\underline{b}) > \varkappa$. Then $Rf(\underline{b},\varkappa)$ iff Rf holds for injective mappings. Analogously for Rfip.

(iv) Evidently $Rf(\underline{b},\varkappa)$ iff $Rf(Comp(\underline{b}),\varkappa)$ and $Rfip(Comp(\underline{b}),\varkappa)$ implies $Rfip(\underline{b},\varkappa)$.

First we shall pay our attention to the existence of refining systems of a given system.

1.3. LEMMA. Assume that $\underline{b}$ is a Boolean algebra and $u:\varkappa \rightarrow b^+$. Then (a) $\rightarrow$ (b) where

(a) There is a partition p of $\underline{b}$ such that $\alpha \in \varkappa$ implies $|u_\alpha \wedge\wedge\ p| \geq \varkappa$.

(b) Rf(u).

We give a standard construction of a refining system which will be called the refining system generated by the partition p.

Proof. For $\alpha \in \varkappa$ let $p_\alpha = \{x \in p;\ x \wedge u_\alpha \neq \mathbb{O}\}$. Then there exists an injective mapping $w: \varkappa \rightarrow p$ such that $w(\alpha) \in p_\alpha$.
Put $v(\alpha) = u(\alpha) \wedge w(\alpha)$. Then v is a disjoint refining system for u.

The next theorem is a generalization of a theorem contained in [1] and [2].

1.4. THEOREM. Assume that $\underline{b}$ is a Boolean algebra and $\varkappa$ is an infinite cardinal. If $sat(\underline{b}) > \varkappa^+$ then $Rf(\underline{b},\varkappa)$.

Proof. Let $u: \varkappa \rightarrow b^+$. By transfinite recursion, we construct a partition p which fulfils (a) of Lemma 1.3.

Step 0. Let p_0 be a partition of $\underline{b}_{u_0}$, $|p_0| = \varkappa^+$. Put
$S_0 = \{\alpha \in \varkappa\ ; |u_\alpha \wedge\wedge\ p_0| = \varkappa^+\}$ and
$\bar{p}_0 = \{x \in p_0\ ;\ (\exists \beta \in \varkappa - S_0)(u_\beta \wedge x \neq \mathbb{O})\}$. Since $|\bar{p}_0| \leq \varkappa$, we have $|u_\alpha \wedge\wedge (p_0 - \bar{p}_0)| = \varkappa^+$ for each $\alpha \in S_0$; furthermore $0 \in S_0$.

Step 1. $\alpha \in \varkappa$, $\alpha > 0$. If we put

$R_\alpha = \bigcup \{S_\beta\ ; \beta \in \alpha\ \&\ S_\beta \subseteq \varkappa\}$ and

$q_\alpha = \bigcup \{p_\beta - \bar{p}_\beta\ ;\ \beta \in \alpha\ \&\ (\gamma \in S_\beta \rightarrow |u_\gamma \wedge\wedge\ (p_\beta - \bar{p}_\beta)| = \varkappa^+)\}$

then q_α is a partition, $u_\gamma \wedge x = \mathbb{O}$ for $\gamma \in \varkappa - R_\alpha$, $x \in q_\alpha$.
Moreover, $\alpha \subseteq R_\alpha$.

Suppose $\varkappa - R_\alpha \neq 0$, since otherwise the partition q_α fulfils (a) of Lemma 1.3 and the proof is finished. Let p_α be a partition of $\underline{b}_{u_\gamma}$ where $\gamma = \min(\varkappa - R_\alpha)$, $|p_\alpha| = \varkappa^+$ and put
$S_\alpha = \{\beta \in \varkappa \; ; \; |u_\beta \wedge\wedge \; p_\alpha| = \varkappa^+\}$ and
$\bar{p}_\alpha = \{x \in p_\alpha \; ; \; (\exists \beta \in \varkappa - S_\alpha)(u_\beta \wedge x \neq \mathbb{O})\}$. Then $|\bar{p}_\alpha| \leq \varkappa$ and for $\beta \in S_\alpha$ we have $|u_\beta \wedge\wedge |p_\alpha - \bar{p}_\alpha|) = \varkappa^+$; furthermore $\alpha \in (S_\alpha \cup R_\alpha)$. Let $\delta = \sup\{\alpha + 1 \; ; \; \varkappa - R_\alpha \neq 0\}$ and $p = \bigcup\{p_\alpha - \bar{p}_\alpha \; ; \; \alpha \in \delta\}$. Then $\delta \leq \varkappa$ and for all $\alpha \in \varkappa$ we have $|u(\alpha) \wedge\wedge \; p| = \varkappa^+$.

This theorem gives us the best result concerning Rf with respect to saturatedness. Boolean algebras with a basis of cardinality $\varkappa$ do not have disjoint refinement property for systems of cardinality $\varkappa$.

1.5. The following definition appears in [1].

DEFINITION. Let $\underline{b}$ be a Boolean algebra and let $u : \varkappa \longrightarrow b^+$

(i) Nd(u) holds if there exists a $v_0 \in b^+$ such that $v_0 \leq u_0$ and for each $\alpha > 0$ we have $u(\alpha) - v_0 \neq \mathbb{O}$.

(ii) We say that $\underline{b}$ has a nowhere dense set of cardinality $\varkappa$, in symbols $Nd(\underline{b},\varkappa)$ if every $u : \varkappa \longrightarrow b^+$ satisfies Nd(u).

Observe that $Nd(\underline{b},\varkappa)$ implies that algebra $\underline{b}$ is atomless.

LEMMA. A Boolean algebra $\underline{b}$ has the property $Nd(\underline{b},\varkappa)$ iff $(\forall x \in b^+)$ ($\underline{b}_x$ has no basis of cardinality $\leq \varkappa$).

Proof. Let $\{u(\alpha) \; ; \; \alpha < \lambda\}$ be a basis of $\underline{b}_x$ for some $x \in b^+$ and $\lambda \leq \varkappa$. Suppose $Nd(\underline{b},\varkappa)$. Then for every $v_0 \in b_x^+$ there is an $\alpha < \lambda$, $\alpha > 0$ such that $u_\alpha \leq v_0$ i.e. $u_\alpha - v_0 = \mathbb{O}$ which contradicts to $Nd(\underline{b},\varkappa)$. Let $u : \varkappa \longrightarrow b^+$ and $\neg\, Nd(u)$; then for every $v_0 \leq u_0$ and $v_0 \neq \mathbb{O}$ there is an $\alpha \in \varkappa$ such that $u_\alpha - v_0 = \mathbb{O}$ i.e. $u_\alpha \leq v_0$. This means that $\{u(\alpha) \; ; \; \alpha \in \varkappa\} \cap b_{u_0}$ is a basis of $\underline{b}_{u_0}$.

1.6. DEFINITION. Let $\underline{b}$ be a Boolean algebra and let $\varkappa$, λ be infinite cardinals. We say that $\underline{b}$ satisfies the $(\aleph_0,\lambda,\varkappa)$ nondistributivity law, or that $\underline{b}$ is $(\aleph_0,\lambda,\varkappa)$-nondistributive if there exists a system $\{p_n \; ; \; n \in \omega\}$ of maximal partitions of $\underline{b}$ such that $(\forall n \in \omega)(|p_n| \leq \lambda)$ and

$(\forall x \in b^+)(\exists n \in \omega)(|p_n \wedge\wedge x| \geq \varkappa)$. $\underline{b}$ is $(\aleph_0, \cdot, \varkappa)$ nondistributive if there exists λ such that $\underline{b}$ is $(\aleph_0, \lambda, \varkappa)$ nondistributive.

If $\underline{b}$ is $(\aleph_0, \cdot, \varkappa)$ nondistributive then obviously $\mathrm{sat}(\underline{b}) > \varkappa$. Moreover, a system $\{p_n ; n \in \omega\}$, which exemplifies the nondistributivity of $\underline{b}$ can be chosen in such a way that p_{n+1} refines p_n and $(\forall x \in p_n)(|\{y ; y \in p_{n+1} \,\&\, y \leq x\}| \geq \varkappa)$.

THEOREM. Assume that $\underline{b}$ is a $(\aleph_0, \cdot, \varkappa)$-nondistributive complete Boolean algebra. Then the following conditions are equivalent:

(i) $\mathrm{Nd}(\underline{b}, \varkappa)$

(ii) $\mathrm{Rf}(\underline{b}, \varkappa)$

(iii) $(\forall x \in b^+)(\underline{b}_x$ is not isomorphic to $\mathrm{Col}(\omega, \varkappa))$.

We shall break the proof into several lemmas.

1.7. LEMMA. Under the assumption of Theorem 1.6., $\mathrm{Nd}(\underline{b}, \varkappa)$ implies $\mathrm{Rf}(\underline{b}, \varkappa)$.

Proof. Let $\{p_n ; n \in \omega\}$ be a given $(\aleph_0, \lambda, \varkappa)$-nondistributive system for $\underline{b}$, and suppose that p_{n+1} is a refinement of p_n. Let $u : \varkappa \rightarrow b^+$. Put

$$q_0 = \{\alpha \in \varkappa ; |u_\alpha \wedge\wedge p_0| \geq \varkappa\}, \text{ and for } n > 0 \text{ put}$$

$$q_n = \{\alpha \in \varkappa ; |u_\alpha \wedge\wedge p_n| \geq \varkappa \,\&\, |u_\alpha \wedge\wedge p_{n-1}| < \varkappa\}.$$

The system $\{q_n ; n \in \omega\}$ is pairwise disjoint and $\bigcup\{q_n, n \in \omega\} = \varkappa$. Let $z_n : q_n \rightarrow b^+$ be a refining system for $u|q_n$ generated by p_n, see Lemma 1.3. Since p_{n+1} is a refinement of p_n we have

(1) $(z_n(\alpha) \wedge z_m(\gamma) \neq \mathbb{O} \,\&\, z_n(\beta) \wedge z_m(\gamma) \neq \mathbb{O}) \rightarrow \alpha = \beta$

for $n < m$ and $\alpha, \beta \in q_n$, $\gamma \in q_m$.

Let $z = \bigcup\{z_n ; n \in \omega\}$. For $\alpha \in \varkappa$ we use $\mathrm{Nd}(\underline{b}, \varkappa)$ on the system $\bar{z}$, where $\bar{z}(0) = z(\alpha)$, $\bar{z}(\alpha) = z(0)$ and $\bar{z}(\beta) = z(\beta)$ for $\beta \neq 0, \alpha$. We obtain a $v(\alpha)$, $\mathbb{O} \neq v(\alpha) \leq z(\alpha)$ such that $\beta \neq \alpha$ implies $z(\beta) - v(\alpha) \neq \mathbb{O}$. We proceed in the construction of a refining system $\{v_\alpha ; \alpha \in \varkappa\}$ by recursion.

Step 0. For $\alpha \in q_0$ we have $v(\alpha)$; $w_0 = \{v(\alpha) ; \alpha \in q_0\}$ is a pairwise disjoint system. As (1) we have $z(\beta) - \bigvee w_0 \neq \mathbb{O}$ for $n > 0$, $\beta \in q_n$. For $n > 0$, $\alpha \in q_n$ put $z_n^{(1)}(\alpha) = z(\alpha) - \bigvee w_0$ Clearly $z_n^{(1)}$ refines z_n.

Step j+1. We have

(i) a disjoint system $\{v(\alpha) ; \alpha \in \bigcup\{q_k ; k \leq j\}\}$ and

(ii) a system of mappings $\{z_n^{(j+1)} ; n \geq j+1\}$ such that for every $\alpha \in \bigcup\{q_k ; k \leq j\}$, $n \geq j+1$ and $\beta \in q_n$, $v(\alpha) \leq z(\alpha)$ and $z_n^{(j+1)}$ refines z_n and $z_n^{(j+1)}(\beta)$ is disjoint with every $v(\alpha)$.

For every $\alpha \in q_{j+1}$ we use Nd on the element $z_{j+1}^{(j+1)}(\alpha)$ according to the system $\bigcup\{z_n^{(j+1)}; n \geq j+1\}$ and obtain $w_{j+1} = \{v(\alpha) ; \alpha \in q_{j+1}\}$ such that $\mathbb{O} \neq v(\alpha) \leq z_{j+1}^{(j+1)}(\alpha)$, $z_n^{(j+1)}(\beta) - \bigvee w_{j+1} \neq \mathbb{O}$ for $n > j+1$, and $\beta \in q_n$. For $n \geq j+2$ and $\beta \in q_n$ put

$$z_n^{(j+2)}(\beta) = z_n^{(j+1)}(\beta) - \bigvee w_{j+1}.$$

The system $v = \{v(\alpha) ; \alpha \in \varkappa\}$ thus obtained is a refining one for u.

1.8. The implication (ii) $\rightarrow$ (iii) in Theorem 1.6 is obvious because the Boolean algebra $Col(\omega,\varkappa)$ is atomless and has a basis of cardinality $\varkappa$.

LEMMA (Mc Aloon). Assume $\underline{b}$ is a complete Boolean algebra. Then the following statements are equivalent.

(i) $\underline{b}$ is isomorphic to $Col(\omega,\varkappa)$;

(ii) $\underline{b}$ is $(\aleph_0, \cdot ,\varkappa)$-nondistributive and has basis of cardinality $\varkappa$.

Proof. (i) $\rightarrow$ (ii) is clear. Let us prove (ii) $\rightarrow$ (i). Let u be a basis of $\underline{b}$, $|u| = \varkappa$. Then every partition p of $\underline{b}$ has the cardinality at most $\varkappa$. Hence $\underline{b}$ is $(\aleph_0,\varkappa,\varkappa)$-nondistributive. Let $\{p_n ; n \in \omega\}$ be a $(\aleph_0,\varkappa,\varkappa)$ nondistributive system. Put

$u_0 = \{x \in u ; |x \wedge\wedge p_0| = \varkappa\}$ and for $n > 0$ put

$u_n = \{x \in u ; |x \wedge\wedge p_n| = \varkappa \,\&\, |x \wedge\wedge p_{n-1}| < \varkappa\}$.

For every n there is an injective mapping $f_n : u_n \rightarrow p_n$ such that $f_n(x) \wedge x \neq \mathbb{O}$. Put

$$\bar{p}_n = \{y ; (y \in p_n \,\&\, y \notin W(f_n)) \vee (\exists x \in u_n)(y \neq \mathbb{O} \,\&\, (y = x \wedge f_n(x) \vee y = -x \wedge f_n(x)))\}.$$

Then $\bigcup\{\bar{p}_n ; n \in \omega\}$ is a basis of $\underline{b}$. Choose a $v : b^+ \rightarrow P(b^+)$ such that $|v(x)| = \varkappa$, $v(x)$ is a maximal partition of $\underline{b}_x$.

Such v certainly exists. We construct $\{q_n ; n \in \omega\}$ by recursion as follows:

(step 0) $q_0 = \bigcup\{v(x) ; x \in \bar{p}_0\}$.

(step n+1) $q_{n+1} = \bigcup\{v(x) ; x \in \bar{p}_{n+1} \wedge\wedge q_n\}$.

The set $\bigcup\{q_n ; n \in \omega\}$ is a basis of b and is isomorphic to the basis d of $Col(\omega,\varkappa)$ (see section 0) with respect to the canonical orderings of algebras.

By proving Lemma 1.8 we have finished the proof of Theorem 1.6 because we have proved (i) → (ii), (ii) → (iii), (iii) → (i).

1.9. The theorem, we just proved says that, roughly speaking, for $(\aleph_0, \cdot, \varkappa)$-nondistributive Boolean algebras, $\neg Rf(\varkappa)$ is equivalent to the existence of a basis of cardinality $\varkappa$.
The $(\aleph_0, \cdot, \varkappa)$-nondistributivity is essential. There are examples of Boolean algebras with saturation equal to $\varkappa^+$, which do not satisfy $Rf(\varkappa)$ and which, furthermore, have no local basis of cardinality less then $\varkappa^+$.

Now we turn our attention to the property Rfip. We offer no definitive results, only some conjectures and propositions. There are examples of complicated algebras with a basis of cardinality $\varkappa$, but not complete, which satisfy $Rfip(\varkappa)$. For example if GCH then the factor algebra of $P(\varkappa)$ modulo the ideal of sets of cardinality less than $\varkappa$ fulfils Rfip (See [5] and [9]).

The main question that we cannot answer goes like this
(A) Does there exist a complete Boolean algebra with a basis of cardinality $\varkappa > \aleph_0$ which satisfies $Rfip(\varkappa)$?

1.10. The above problem can be reduced to the algebra $Col(\omega,\varkappa)$.

PROPOSITION. Assume $\varkappa$ is an uncountable cardinal. If there is a complete Boolean algebra with a basis of cardinality $\varkappa$ with the property $Rfip(\varkappa)$ then the Boolean algebra $Col(\omega,\varkappa)$ has the property $Rfip(\varkappa)$, too.

Proof. Assume b is a complete Boolean algebra with a basis of cardinality $\varkappa$ and $Rfip(b,\varkappa)$. According to a slight generalization of Kripke's embedding theorem we can assume that b is a complete subalgebra of $Col(\omega,\varkappa)$. This well known fact can be proved by using the characterization of the algebra $Col(\omega,\varkappa)$ (Lemma 1.8) and properties of free products of Boolean algebras. Let $u = \{u_\alpha ; \alpha \in \varkappa\}$ be a system of nonzero elements of Col $(\omega, \varkappa)$ with the finite intersection property. Put $u^{\#}_\alpha = \bigwedge\{x \in b \; ; \; x \geq u_\alpha\}$. The system $u^{\#} = \{u^{\#}_\alpha ; \alpha \in \varkappa\}$ has the finite intersection property, too. Let $w = \{w_\alpha \; ; \; \alpha \in \varkappa\}$ be a refining system for $u^{\#}$ in the algebra $\underline{b}$. If we put $v(\alpha) = w(\alpha) \wedge u(\alpha)$ then $v(\alpha) \neq \mathbb{O}$ and the system $v = \{v_\alpha \; ; \; \alpha \in \varkappa\}$ refines u.

1.11. The following propositions give some relations between properties of ultrafilters and filters on a Boolean algebra and the property Rfip.

THEOREM. Assume $\underline{b}$ is a complete Boolean algebra and $\varkappa$ is an infinite cardinal. Then (a) $\rightarrow$ (b) $\rightarrow$ (c) where

(a) for any ultrafilter j on $\underline{b}$ there is a maximal partition p of $\underline{b}$ such that $(\forall p_0 \subseteq p)(|p_0| < \varkappa \rightarrow \bigvee p_0 \notin j)$;

(b) Rfip$(\underline{b}, \varkappa)$;

(c) no ultrafilter on $\underline{b}$ has a basis of cardinality at most $\varkappa$.

Moreover if $\varkappa = \sum_{\lambda < \varkappa} \varkappa^\lambda$ and $\underline{b}$ has a basis of cardinality $\varkappa$ then the above conditions are equivalent.

Proof. (a) $\rightarrow$ (b). For any $u : \varkappa \rightarrow b^+$ with the finite intersection property take an ultrafilter j such that $u(\alpha) \in j$ for all $\alpha \in \varkappa$. Let p be a maximal partition guaranteed by (a). The system u and the partition p fulfil condition of Lemma 1.3.

(b) $\rightarrow$ (c). Assume, on the contrary, that $u = \{u_\alpha \; ; \; \alpha < \lambda\}$, $\lambda \leq \varkappa$, is a basis of an ultrafilter j and $v = \{v_\alpha \; ; \; \alpha < \lambda\}$ is a disjoint refining system for u. Let p be a maximal partition of $\underline{b}$ which contains v, i.e. $v \subseteq p$. From Rfip$(\varkappa)$ it follows that $\underline{b}$ is atomless, so we can split every element of p into two nonzero elements. Choose a system $\{(x_1, x_2) \; ; \; x \in p\}$ such that $x_1, x_2 \neq \mathbb{O}$, $x_1 \wedge x_2 = \mathbb{O}$ & $x_1 \vee x_2 = x$ for any $x \in p$. Put $a_1 = \bigvee\{x_1 \; ; \; x \in p\}$, $a_2 = \bigvee\{x_2 \; ; \; x \in p\}$. Clearly $a_1 = -a_2$ but no v_α lies under a_1 or a_2, which is a contradiction.

(c) $\rightarrow$ (a). In fact we show $\neg$(a) $\rightarrow$ $\neg$(c). Let $u = \{u_\alpha ; \alpha \in \varkappa\}$ be a basis of $\underline{b}$. Let j be an ultrafilter with property $\neg$(a). We construct a basis of j of cardinality $\varkappa$. For $x \in j$ take a maximal partition p_x of $\underline{b}_x$ such that p_x contains only elements of the basis u. Then there exists $q_x \subseteq p_x$, $|q_x| < \varkappa$ such that $\bigvee q_x \in j$. Hence

$w = \{\bigvee q_x \ ; \ x \in j\}$ is a basis of the ultrafilter j and it has a cardinality at most $\varkappa$.

1.12. The following proposition is motivated by the characterization given by Prikry in [7].

PROPOSITION. Assume $\varkappa$ is a regular cardinal, $\underline{b}$ is a complete Boolean algebra. Then the following conditions are equivalent.

(i) $\mathrm{Rfip}(\underline{b},\varkappa)$;

(ii) for every filter j on $\underline{b}$ with a basis of cardinality at most $\varkappa$ there is a $\varkappa$-complete filter F such that $\bigwedge F = 0$ and $j \cup F$ has the finite intersection property.

Moreover if $\sum_{\lambda < \varkappa} \varkappa^{\lambda} = \varkappa$ and if $\underline{b}$ has a basis of cardinality $\varkappa$ then (i) is equivalent to

(iii) every ultrafilter j on $\underline{b}$ contains a $\varkappa$-complete filter F such that $\bigwedge F = 0$.

Proof. (i) $\to$ (ii). Assume $u = \{u_\alpha \ ; \ \alpha \in \lambda\}$, $\lambda \leq \varkappa$, is basis of j and let $v = \{v_\alpha \ ; \ \alpha < \lambda\}$ be a disjoint refinement of u. As $\mathrm{sat}(\underline{b}) \geq \varkappa^+$ we can choose for every $\alpha < \lambda$ a partition w_α of $\underline{b}v_\alpha$ such that $w_\alpha \subseteq b^+$, $|w_\alpha| = \varkappa$. Take a maximal partition p of $\underline{b}$ such that $\bigcup\{w_\alpha \ ; \ \alpha \in \varkappa\} \subseteq p$. Put $f = \{-\bigvee p_0 \ ; \ p_0 \subseteq p \ \& \ |p_0| < \varkappa\}$. The family f generates a $\varkappa$-complete filter F. Clearly $\bigwedge F = 0$ and $j \cup F$ has Fip.

(ii) $\to$ (i). Assume $u: \varkappa \to b^+$ has Fip. Let j be the filter generated by u. Let F be a $\varkappa$-complete filter as in (ii). Because $\bigwedge F = 0$ the dual ideal $J = \{-x \ ; \ x \in F\}$ is a basis of $\underline{b}$. Let p be a maximal partition of $\underline{b}$ with elements from J. As J is $\varkappa$-complete ideal every u_α meets p in $\varkappa$ elements. Thus u satisfies the sufficient condition for the existence of a refinement, given in Lemma 1.3.

Let us prove (iii) $\to$ (ii). Let j be a filter on $\underline{b}$. Take an ultrafilter $j_1 \supseteq j$. Let F be a $\varkappa$-complete filter which exists for j_1 by (iii). Clearly $j \cup F$ has the finite intersection property.

(ii) $\to$ (iii). (ii) implies $\mathrm{Rfip}(\underline{b},\varkappa)$. Owing to theorem 1.11 there exists a maximal partition p of $\underline{b}$ such that

$(\forall p_0 \subseteq p)(|p_0| < \varkappa \to \bigvee p_0 \notin j)$. But then $\{-\bigvee p_0 ; p_0 \subseteq p \ \& \ |p_0| < \varkappa\}$ generates a $\varkappa$-complete filter $F \subseteq j$ which obviously satisfies $\bigwedge F = 0$.

2. Conections with a problem of Fodor and a problem of Ulam on families of measures for $\mathcal{K} = \aleph_1$.

2.1. In this section we turn our attention to systems of subsets of $\mathcal{K} = \omega_1$. We shall deal with σ-complete ideals on $P(\omega_1)$ and corresponding factor algebras. Our aim is to show that some hypotheses concerning refining properties are naturally related to a problem of Fodor (see [1]), and furthermore, to a problem of Ulam (see [10]). We shall add one more problem and relate it to refining properties.

Fodor's problem reads as follows: Let i be a σ-complete nontrivial ideal on ω_1. Does Sierpiński's theorem, mentioned in the introduction, hold if "of cardinality $\aleph_1$" is replaced by "not belonging to i" ? In other words, prove or disprove the following statement:

(F) For every σ-complete nontrivial ideal i on ω_1 and every system $u = \{u_\alpha ; \alpha \in \omega_1\} \subseteq P(\omega_1) - i$ there is a disjoint refining system $v = \{v_\alpha ; \alpha \in \omega_1\} \subseteq P(\omega_1) - i$ for u.

The following will be called Ulam's problem. Prove or disprove the following statement:

(U) There exists a family $\{\mu_\alpha ; \alpha \in \omega_1\}$ of σ-additive 0-1 nontrivial measures on ω_1 such that every subset is measurable with respect to one of them.

In addition let us formulate our third problem. Prove or disprove the following statement:

(III) There exists a σ-complete nontrivial ideal i on ω_1 and a set $X \subseteq P(\omega_1)$ of cardinality $\aleph_1$ such that $i \cup X$ generates a maximal ideal.

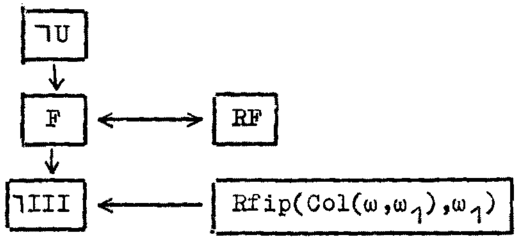
It is known that V = L implies an affirmative answer to Fodor's problem and the negative solution of both Ulam's and the third problem. This can be found in [15],[6],[4].

Nothing is known concerning the consistency of the negative solution of Fodor's problem and the positive solution of Ulam's and the third problem. Thus we conjecture that ZFC proves (F), $\neg$(U) and $\neg$(III).

We shall show some interdependences between (F),(U) and (III) and relate them to the refining properties.

Let (RF) means: For every σ-complete nontrivial ideal i on ω_1 the Boolean algebras $P(\omega_1)/i$ and Col (ω,ω_1) are not isomorphic.

Our results can be summarized in the following figure:



Arrows mean implications.

The following problem arises naturally.

(B) Does (F) $\longrightarrow \neg$ (U) ?

2.2. We begin with some denotations and facts. Assume i is a σ-complete ideal on ω_1. For $u \subseteq \omega_1$ we put $[u] = \{v \subseteq \omega_1 ; (v-u) \cup (u-v) \in i\} \in P(\omega_1)/i$. For $u \in P(\omega_1)-i$ let i_u be the ideal generated by $i \cup \{\omega_1 - u\}$. The ideal i_u is again σ-complete, and if i is nontrivial, so is i_u. The algebra $P(\omega_1)/i_u$ is isomorphic to $(P(\omega_1)/i)_{[u]}$. Furthermore let us define sat(i) = = sat($P(\omega_1)/i$). Remember that our notion of saturation (see Section 0) is, in fact, hereditary saturation.

2.3. It is easy to see that for a σ-complete ideal i and a system $u: \omega_1 \longrightarrow (P(\omega_1)-i)$ the existence of a disjoint refinement for system $\{[u_\alpha] ; \alpha \in \omega_1\}$ in the algebra $P(\omega_1)/i$ is equivalent to the existence of a system $v: \omega_1 \rightarrow (P(\omega_1)-i)$ such that $v_\alpha \subseteq u_\alpha$ and $v_\alpha \cap v_\beta = 0$ for $\alpha \neq \beta$.

2.4. THEOREM (Ulam). Assume i is a σ-complete nontrivial ideal on ω_1. Then $P(\omega_1)/i$ is a $(\aleph_0, \cdot, \aleph_1)$-nondistributive Boolean algebra.

Proof. For $\alpha \in \omega_1$ we define $f_\alpha : \omega_1 \rightarrow \omega_1$ as follows: $f_\alpha(\beta) = \beta$ if $\beta < \alpha$, $f_\alpha(\beta) = \alpha$ if $\beta \geqslant \alpha$. Let f be a mapping from ω_1 to ω_1 whichis "behind" all f_α's , for example $f(\alpha) = \alpha+1$. Let $v_\alpha : f(\alpha) \rightarrow \omega_0$ be injective for any $\alpha \in \omega_1$. Put $g_\alpha(\beta) =$ $= v_\beta(f_\alpha(\beta))$ for $\alpha,\beta \in \omega_1$. Then for $\alpha,\beta \in \omega_1$ we have $g_\alpha : \omega_1 \rightarrow \omega_0$ and $\alpha \neq \beta$ implies $(\exists \gamma_0 \in \omega_1)(\forall \gamma > \gamma_0)(g_\alpha(\gamma) \neq g_\beta(\gamma))$. Every g_α determines a partition on ω_1 as follows:

$c_\alpha^n = \{\beta \in \omega_1 : g_\alpha(\beta) = n\}$. As i is σ-complete it is

$\omega_1 - \bigcup\{c^n_\alpha ; n \in \omega_o \& c^n_\alpha \notin i\} \in i$ for any $\alpha \in \omega_1$. Moreover we have $(\forall n \in \omega)(\forall \alpha,\beta \in \omega_1)(\alpha \neq \beta \rightarrow c^n_\alpha \cap c^n_\beta \in i)$. For $n \in \omega$ set $p_n = \{[c^n_\alpha] ; \alpha \in \omega_1, c^n_\alpha \notin i\}$. Each p_n is a partition on $P(\omega_1)/i$. We show that for any $a \in P(\omega_1) - i$ there exist $n \in \omega$ such that $|[a] \wedge\wedge p_n| = \aleph_1$. Assume not. For $n \in \omega$ let $\alpha_n \in \omega_1$ satisfy $(\forall \alpha > \alpha_n)(a \cap c^n_\alpha \in i)$. Put $\alpha_\omega = \sup\{\alpha_n ; n \in \omega\}$ and choose $\alpha > \alpha_\omega$. Then mapping g_α induce a partition $\{c^n_a ; n \in \omega\}$ on $a \notin i$, where $c^n_a = \{\beta \in a ; g_\alpha(\beta) = n\}$. By the σ-completeness of i, there exists n such that $c^n_a \notin i$. Hence $c^n_a = c^n_\alpha \cap a \notin i$ contradicts the definition of α_n. Now for $n \in \omega$ take a maximal partition q_n of $P(\omega_1)/i$ containing p_n. Then $\{q_n ; n \in \omega\}$ exemplifies the $(\aleph_o, \cdot, \aleph_1)$-nondistributivity of $P(\omega_1)/i$.

2.5. Assume i is nontrivial σ-complete ideal. As a corollary of theorem 2.4. we have that $sat(i) > \omega_1$. If $sat(i) = \omega_2$ then there is an $u \notin i$ such that in the algebra $P(\omega_1)/i_u$ every maximal partition has cardinality at most $\aleph_1$. In this case the algebra $P(\omega_1)/i_u$ is complete (see [14] and [13] p.76).

2.6. THEOREM. (F) $\longleftrightarrow$ (RF).

Proof. F $\rightarrow$ RF. Let i be a nontrivial σ-complete ideal. The algebra $P(\omega_1)/i$ has no base of cardinality $\aleph_1$, hence RF.

$\neg$F $\rightarrow$ $\neg$RF. Let i be a σ-complete nontrivial ideal such that the algebra $P(\omega_1)/i$ has not the property $Rf(\aleph_1)$. Thus $sat(i) = \omega_2$. By the theorems 2.4 and 1.6, even without the completeness of $\underline{b} = P(\omega_1)/i$, we have an element $[u] \in b^+$ such that $\underline{b}_{[u]}$ has a base of cardinality $\aleph_1$. Hence by 2.5 the algebra $P(\omega_1)/i_u$ is complete and isomorphic to the algebra $Col(\omega,\omega_1)$.

2.7. THEOREM. $\neg$U $\rightarrow$ F.

Proof. $\neg$F $\rightarrow$ U. By the theorem 2.6 there is an ideal i such that $P(\omega_1)/i \cong Col(\omega,\omega_1)$. For $\alpha \in \omega_1$ take $x_\alpha \in P(\omega_1) - i$ such that $\{[x_\alpha] ; \alpha \in \omega_1\}$ is a base of $P(\omega_1)/i$. Let i_α be the ideal generated by the set $i \cup \{\omega_1 - x_\alpha\}$. For every $x \subseteq \omega_1$ either $x \in i$ or there exists α such that $(x_\alpha - x) \in i$ and therefore $(\omega_1 - x) \in i_\alpha$.

2.8. The implications (F) $\rightarrow$ $\neg$(III) and $Rfip(Col(\omega,\omega_1),\omega_1)$ $\rightarrow$ $\neg$(III) follow directly from the results in section 1. and the method used in the proof of theorem 2.6.

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