Probabilistic Acceptors for Languages over Infinite Words

Christel Baier
Technical University Dresden

joint work with *Nathalie Bertrand*
and *Marcus Größer*

- languages over infinite words $L \subseteq \Sigma^\omega$
- arise from ω-regular expressions

$$\alpha_1 \beta_1^\omega + \ldots + \alpha_n \beta_n^\omega$$

where α_i, β_i are regular expressions with $\varepsilon \notin \beta_i$

ω–regular languages

- languages over infinite words $L \subseteq \Sigma^\omega$
- arise from ω-regular expressions

$$\alpha_1\beta_1^\omega + \ldots + \alpha_n\beta_n^\omega$$

  where α_i, β_i are regular expressions with $\varepsilon \notin \beta_i$

- recognized by many types of ω-automata, e.g., **NBA (nondeterministic Büchi automata)**

  syntax as **NFA**, but with acceptance criterion

  "visit infinitely often a final state"

- languages over infinite words $L \subseteq \Sigma^\omega$
- arise from ω-regular expressions

$$\alpha_1\beta_1^\omega + \ldots + \alpha_n\beta_n^\omega$$

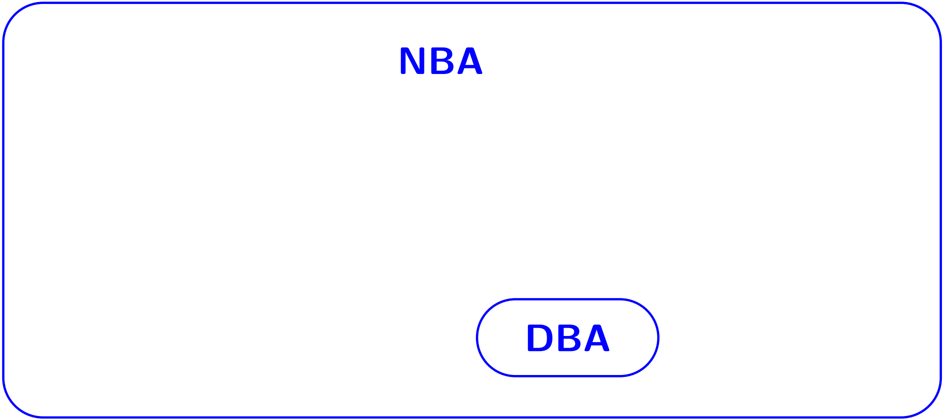
- recognized by many types of ω-automata, e.g., **NBA (nondeterministic Büchi automata)**
- can be used for verifying linear time properties

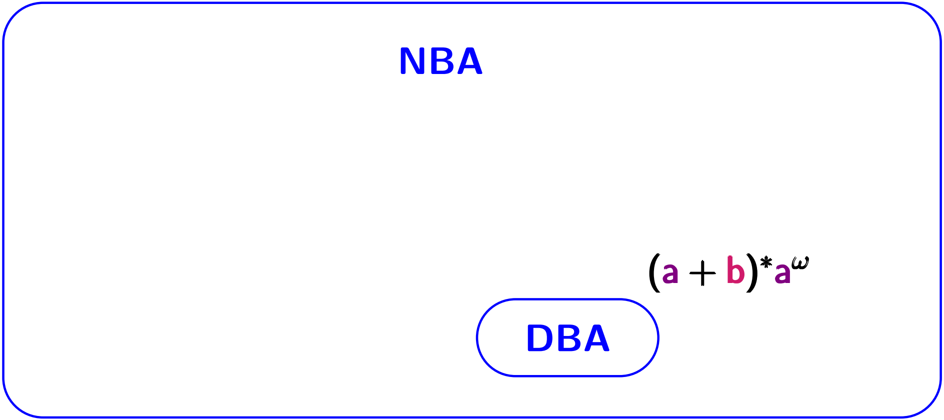
"check whether $\mathcal{L}(\mathcal{M}) \cap \mathcal{L}(\mathcal{A}) \neq \emptyset$ with graph algorithm in the product $\mathcal{M} \times \mathcal{A}$"

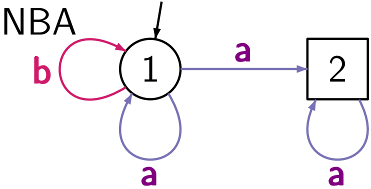
where $\mathcal{M}$ is the **system model**,
$\mathcal{A}$ is an **NBA for the bad behaviors**

NBA

ω-automata




NBA
DBA
(a + b)*a^ω


NBA
b
1
a
2
a
a

ω-automata

NBA $\Box\Diamond F$

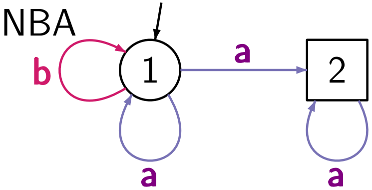
NSA/DSA (Streett automata) $\bigwedge_i (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

NRA/DRA (Rabin automata) $\bigvee_i (\Diamond\Box\neg K_i \wedge \Box\Diamond H_i)$

⋮

$(a+b)^*a^\omega$

DBA



ω-automata

NBA $\Box\Diamond F$

NSA/DSA (Streett automata) $\bigwedge_i (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

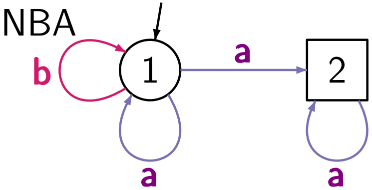
NRA/DRA (Rabin automata) $\bigvee_i (\Diamond\Box\neg K_i \wedge \Box\Diamond H_i)$

⋮

$(a + b)^* a^\omega$

DBA

NBA $\overset{exp}{\rightsquigarrow}$ **DSA/DRA/...**
[Safra '88]



NBA $\Box\Diamond F$

NSA/DSA (Streett automata) $\bigwedge_i (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

NRA/DRA (Rabin automata) $\bigvee_i (\Diamond\Box\neg K_i \wedge \Box\Diamond H_i)$

⋮

probabilistic Büchi automata ?

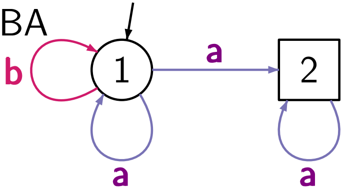
$(a+b)^* a^\omega$

DBA

NBA $\overset{exp}{\rightsquigarrow}$ **DSA/DRA/...**

[Safra '88]

NBA

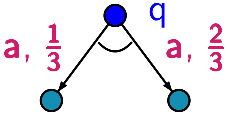


Overview

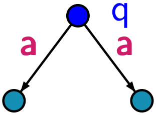
- definition of probabilistic Büchi automata (PBA)
- expressiveness of PBA
- efficiency of PBA
- other acceptance conditions
  (Streett/Rabin, alternative semantics)
- composition operators on PBA
- decision problems for PBA

Probabilistic Büchi Automaton (PBA)

PBA: syntax as for **NBA**,
but all choices are resolved probabilistically



PBA
probabilistic choice



NBA
nondeterministic choice

Probabilistic Büchi Automaton (PBA)

$\mathcal{P} = (Q, \delta, \mu, F)$ over alphabet Σ:

- Q finite state space
- transition probability function
  $\delta : Q \times \Sigma \times Q \to [0,1]$

Probabilistic Büchi Automaton (PBA)

$\mathcal{P} = (Q, \delta, \mu, F)$ over alphabet Σ:

- Q finite state space
- transition probability function

  $\delta : Q \times \Sigma \times Q \to [0, 1]$ s.t.

  for all $q \in Q, a \in \Sigma$: $\sum_{p \in Q} \delta(q, a, p) \in \{0, 1\}$

Probabilistic Büchi Automaton (PBA)

$\mathcal{P} = (Q, \delta, \mu, F)$ over alphabet Σ:

- Q finite state space
- transition probability function
  $\delta : Q \times \Sigma \times Q \to [0,1]$ s.t.
  for all $q \in Q, a \in \Sigma$: $\sum_{p \in Q} \delta(q, a, p) \in \{0, 1\}$
- initial distribution $\mu : Q \to [0,1]$

Probabilistic Büchi Automaton (PBA)

$\mathcal{P} = (Q, \delta, \mu, F)$ over alphabet Σ:

- Q finite state space
- transition probability function
  $\delta : Q \times \Sigma \times Q \to [0,1]$ s.t.

  for all $q \in Q, a \in \Sigma$: $\sum_{p \in Q} \delta(q, a, p) \in \{0,1\}$
- initial distribution $\mu : Q \to [0,1]$
- set of final states $F \subseteq Q$

Semantics of PBA

$\mathcal{P} = (\mathbf{Q}, \delta, \mu, \mathbf{F})$

- $\mathbf{Q}$ finite state space
- transition probability function
  $\delta : \mathbf{Q} \times \mathbf{\Sigma} \times \mathbf{Q} \to [0, 1]$
- initial distribution μ
- set of final states $\mathbf{F} \subseteq \mathbf{Q}$

$\mathcal{P} = (\mathbf{Q}, \delta, \mu, \mathbf{F})$

- $\mathbf{Q}$ finite state space
- transition probability function
  $\delta : \mathbf{Q} \times \boldsymbol{\Sigma} \times \mathbf{Q} \to [0, 1]$
- initial distribution μ
- set of final states $\mathbf{F} \subseteq \mathbf{Q}$

For $\mathbf{x} \in \boldsymbol{\Sigma}^\omega$:

$\mathbf{Pr}(\mathbf{x})$ = probability for the **accepting runs** for $\mathbf{x}$

Semantics of PBA

$$\mathcal{P} = (\mathbf{Q}, \delta, \mu, \mathbf{F})$$

- $\mathbf{Q}$ finite state space
- transition probability function
  $\delta : \mathbf{Q} \times \mathbf{\Sigma} \times \mathbf{Q} \rightarrow [0, 1]$
- initial distribution μ
- set of final states $\mathbf{F} \subseteq \mathbf{Q}$

For $\mathbf{x} \in \mathbf{\Sigma}^\omega$:

$\mathbf{Pr}(\mathbf{x})$ = probability for the **accepting runs** for $\mathbf{x}$

accepting run: visits $\mathbf{F}$ infinitely often

Semantics of PBA

$\mathcal{P} = (\mathbf{Q}, \delta, \mu, \mathbf{F})$ $\longleftarrow$ can be viewed as a *Markov decision process*

- $\mathbf{Q}$ finite state space
- transition probability function
  $\delta : \mathbf{Q} \times \mathbf{\Sigma} \times \mathbf{Q} \to [0, 1]$
- initial distribution μ
- set of final states $\mathbf{F} \subseteq \mathbf{Q}$

For $\mathbf{x} \in \mathbf{\Sigma}^\omega$:

$\mathbf{Pr}(\mathbf{x}) =$ **probability** for the accepting runs for $\mathbf{x}$

probability measure in the infinite Markov chain induced by $\mathbf{x}$ viewed as a scheduler

$\mathcal{P} = (\mathbf{Q}, \delta, \mu, \mathbf{F})$ ⟵ can be viewed as a *Markov decision process*

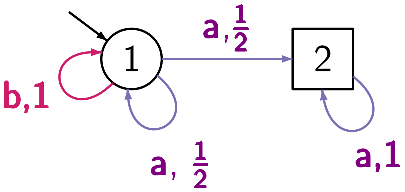
- $\mathbf{Q}$ finite state space
- transition probability function
  $\delta : \mathbf{Q} \times \mathbf{\Sigma} \times \mathbf{Q} \to [0,1]$
- initial distribution μ
- set of final states $\mathbf{F} \subseteq \mathbf{Q}$

For $\mathbf{x} \in \mathbf{\Sigma}^\omega$:

$\mathrm{Pr}(\mathbf{x})$ = probability for the accepting runs for $\mathbf{x}$

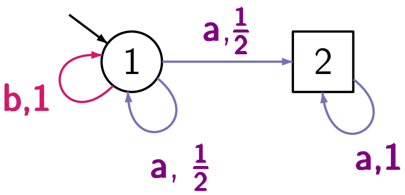
accepted language: $\mathcal{L}(\mathcal{P}) = \{\mathbf{x} \in \mathbf{\Sigma}^\omega : \mathrm{Pr}(\mathbf{x}) > 0\}$

Example for a PBA



- final state
- non-final state

Example for a PBA



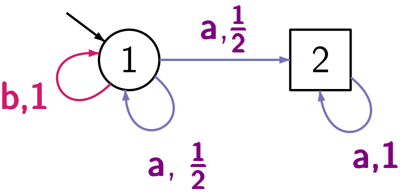
input word

$x = aaba^\omega$

□ final state

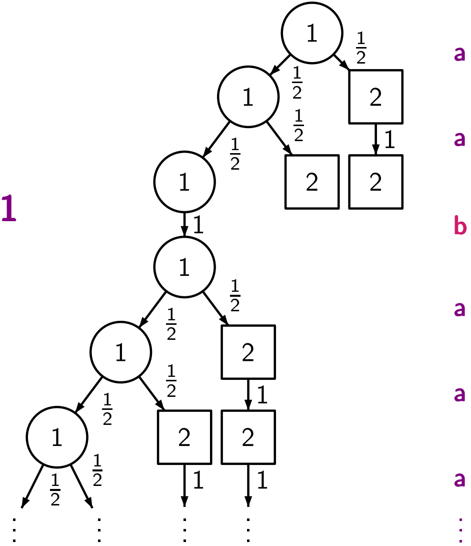
○ non-final state

Example for a PBA

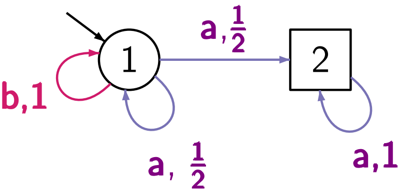


input word

$x = aaba^{\omega}$

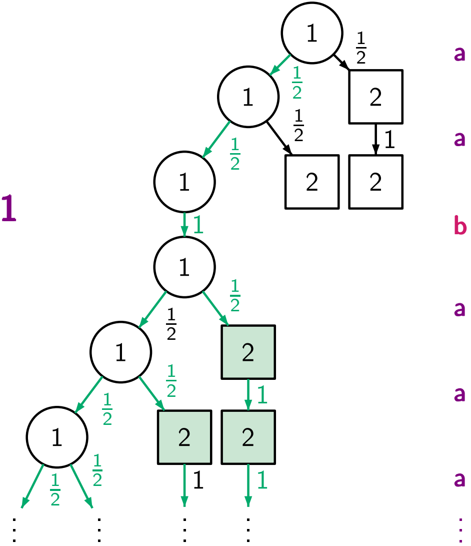


Example for a PBA



input word

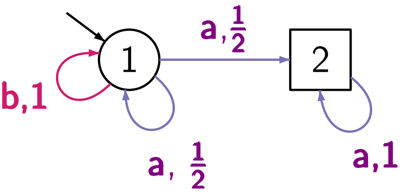
$x = aaba^{\omega}$



$\Pr(x) = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$

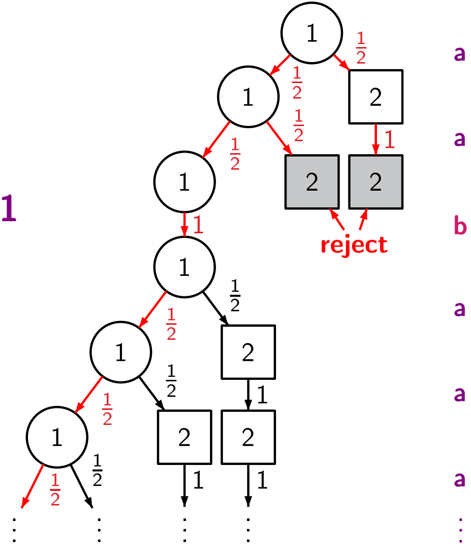
$= \frac{1}{4}$

Example for a PBA



input word

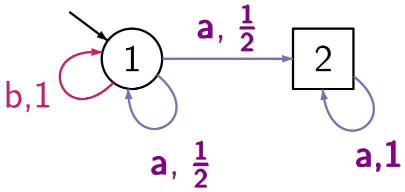
$x = aaba^\omega$



$Pr(x) = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots$

$= 1\text{-}Pr(x \text{ is rejected})$

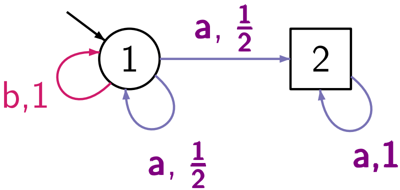
Example for PBA



accepted language:

$(a+b)^*a^\omega$

Example for PBA

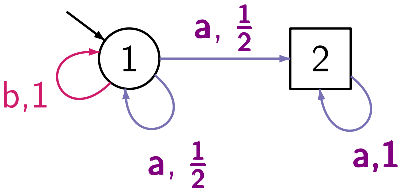


accepted language:

$(a + b)^* a^\omega$

Thus: **PBA** are strictly more expressive than **DBA**

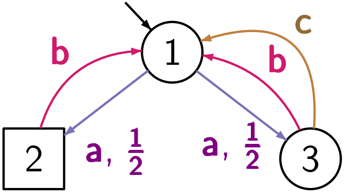
Example for PBA



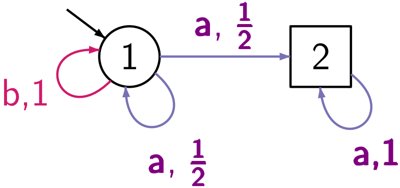
accepted language:

$(a + b)^* a^\omega$

Thus: **PBA** are strictly more expressive than **DBA**



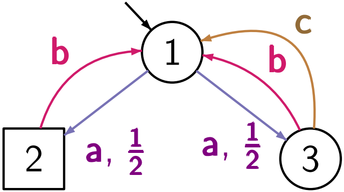
Example for PBA



accepted language:

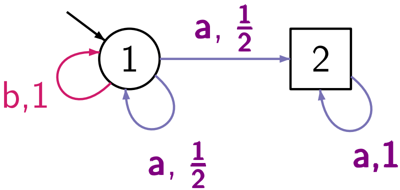
$(a + b)^* a^\omega$

Thus: **PBA** are strictly more expressive than **DBA**



NBA accepts $((ac)^* ab)^\omega$

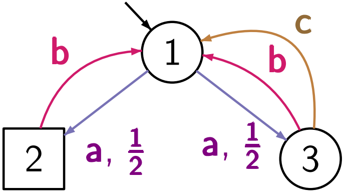
Example for PBA



accepted language:

$(a + b)^* a^\omega$

Thus: **PBA** are strictly more expressive than **DBA**



accepted language:

$(ab + ac)^* (ab)^\omega$

but NBA accepts $((ac)^* ab)^\omega$

Expressiveness of PBA

Expressiveness of PBA

Theorem:
PBA are strictly more expressive than **NBA**

PBA are strictly more expressive than NBA

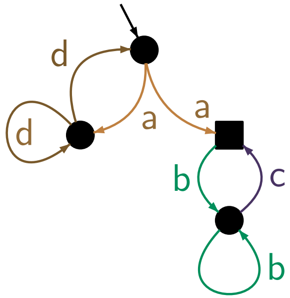
from NBA to PBA:

NBA $\longrightarrow$ (Courcoubetis/ Yannakakis '95) NBA deterministic in limit = PBA

PBA are strictly more expressive than NBA

from NBA to PBA:

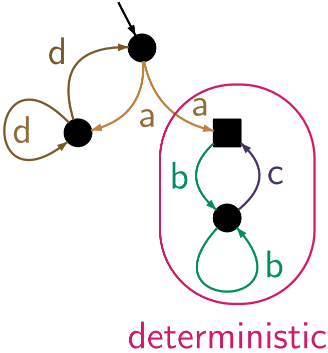
NBA ⟶ (Courcoubetis/ Yannakakis '95) NBA deterministic in limit = PBA



PBA are strictly more expressive than NBA

from NBA to PBA:

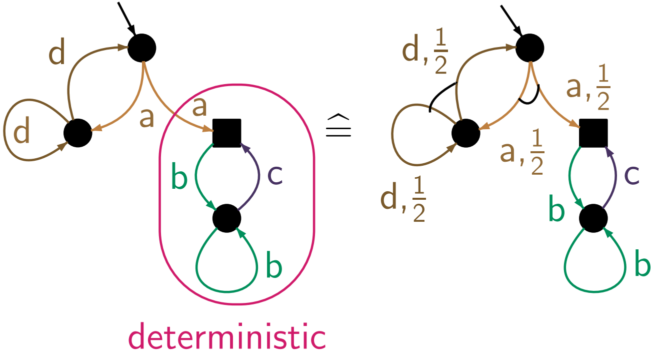
NBA ⟶ (Courcoubetis/ Yannakakis '95) NBA deterministic in limit = PBA



PBA are strictly more expressive than NBA

from NBA to PBA:

NBA ⟶ (Courcoubetis/Yannakakis '95) NBA deterministic in limit $=$ PBA



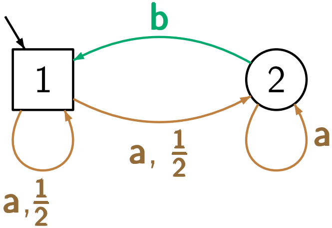
PBA are strictly more expressive than **NBA**

- from NBA to PBA:
  via NBA that are deterministic in limit
- PBA can accept non-ω-regular languages

PBA are strictly more expressive than **NBA**

- from NBA to PBA:
  via NBA that are deterministic in limit

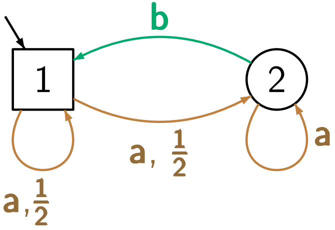
- PBA can accept non-ω-regular languages



PBA are strictly more expressive than NBA

- from NBA to PBA:
  via NBA that are deterministic in limit

- PBA can accept non-ω-regular languages

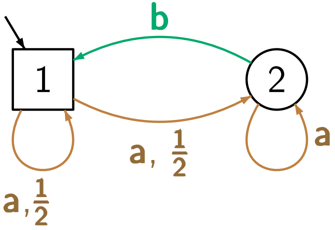


accepted language:

$$\left\{ a^{k_1} b a^{k_2} b a^{k_3} b \ldots : \qquad \qquad \qquad \qquad \right\}$$

PBA are strictly more expressive than NBA

- from NBA to PBA:
  via NBA that are deterministic in limit
- PBA can accept non-ω-regular languages

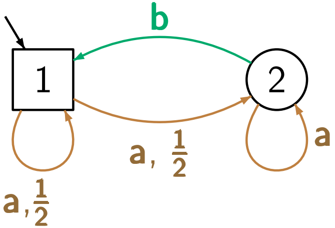


accepted language:

$$\left\{ a^{k_1} b a^{k_2} b a^{k_3} b \ldots : \prod_{i=1}^{\infty} \left(1 - \left(\frac{1}{2}\right)^{k_i}\right) > 0 \right\}$$

PBA are strictly more expressive than NBA

- from NBA to PBA:
  via NBA that are deterministic in limit
- PBA can accept non-ω-regular languages

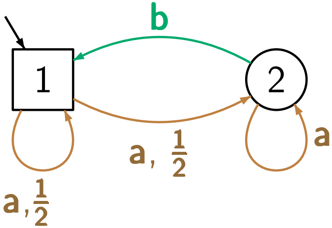


- uniform PBA cover exactly the class of ω-regular languages

PBA are strictly more expressive than **NBA**

- from NBA to PBA:
  via NBA that are deterministic in limit

- PBA can accept non-ω-regular languages



- uniform PBA cover exactly the class of ω-regular languages where "uniformity" is a probabilistic condition on end components

Efficiency of PBA

Efficiency of PBA

Theorem:

There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$ s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

NSA: nondeterministic Streett automaton

Efficiency of PBA

Theorem:
There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$ s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

Proof: consider the ω-regular languages

$$L_n \; = \; \left\{ xy^\omega \; : \; x, y \in \{a, b\}^*, |y| = n \right\}$$

Efficiency of PBA

Theorem:

There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$ s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

Proof: consider the ω-regular languages

$$L_n = \{ xy^\omega : x, y \in \{a, b\}^*, |y| = n \}$$

show that

1. Each any NSA for L_n has $\geq 2^n/n$ states.
2. There is a uniform PBA with $\mathcal{O}(n)$ states.

Efficiency of PBA

Theorem:

There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$ s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

Proof: consider the ω-regular languages

$$L_n = \{ xy^\omega : x, y \in \{a, b\}^*, |y| = n \}$$

Each any NSA for L_n has $\geq 2^n/n$ states

Theorem:

There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$ s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

Proof: consider the ω-regular languages

$$L_n = \{ xy^\omega : x, y \in \{a, b\}^*, |y| = n \}$$

Each any NSA for L_n has $\geq 2^n/n$ states

as the "accepting loops" for $y_1 \not\equiv y_2$, $|y_1| = |y_2| = n$ are disjoint

where for $y_1 = c_1c_2\ldots c_n$ and $y_2 = d_1d_2\ldots d_n$:

$$y_1 \equiv y_2 \text{ iff } \exists i \text{ s.t. } c_1c_2\ldots c_n = d_i\ldots d_nd_1\ldots d_{i-1}$$

Efficiency of PBA

Thm: There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$ s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

Proof: consider the ω-regular languages

$$\mathsf{L}_n \;=\; \big\{\, xy^\omega \;:\; x, y \in \{a, b\}^*, |y| = n \,\big\}$$

show that

1. each NSA for L_n has $\geq 2^n/n$ states $\checkmark$
2. there exists a uniform PBA with $2n$ states

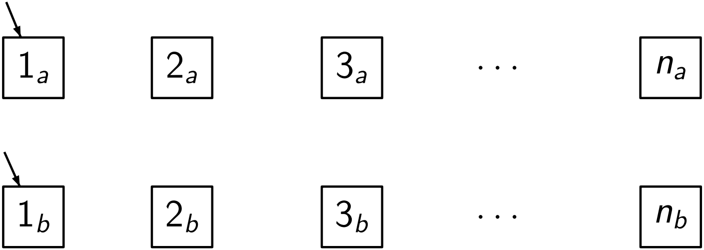
Efficiency of PBA

Thm: There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$
s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

Proof: consider the ω-regular languages

$$L_n = \{ xy^\omega : x, y \in \{a, b\}^*, |y| = n \}$$

PBA for L_n mit $2n$ states:



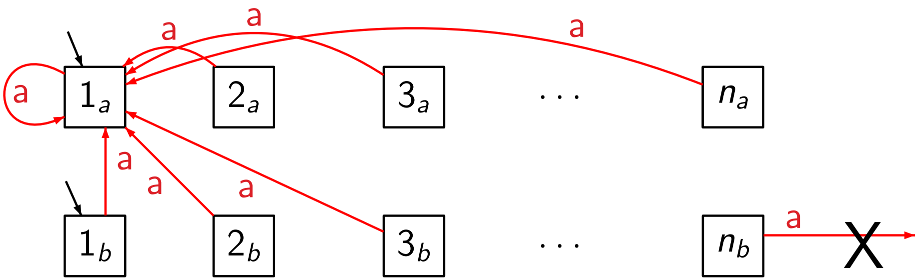
Efficiency of PBA

Thm: There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$
s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

Proof: consider the ω-regular languages

$$L_n = \{ xy^\omega : x, y \in \{a, b\}^*, |y| = n \}$$

PBA for L_n mit $2n$ states:



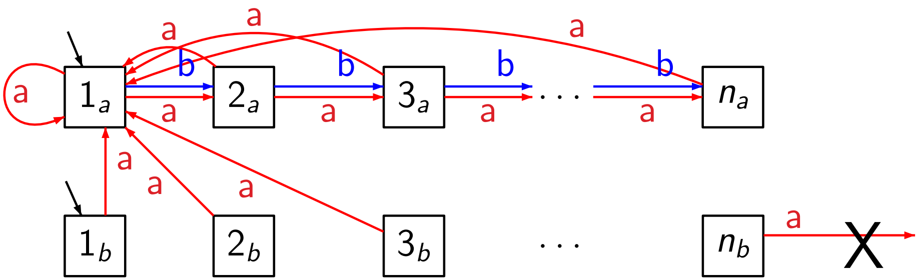
Efficiency of PBA

Thm: There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$
s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

Proof: consider the ω-regular languages

$$\mathsf{L}_n = \{ xy^\omega : x, y \in \{a, b\}^*, |y| = n \}$$

PBA for L_n mit $2n$ states:



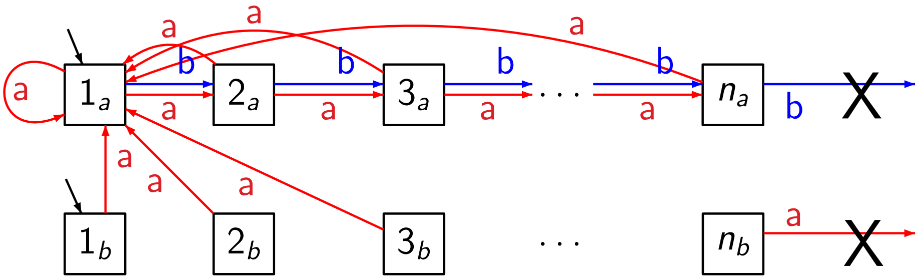
Efficiency of PBA

Thm: There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$
s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

Proof: consider the ω-regular languages

$$L_n = \{ xy^\omega : x, y \in \{a, b\}^*, |y| = n \}$$

PBA for L_n mit $2n$ states:



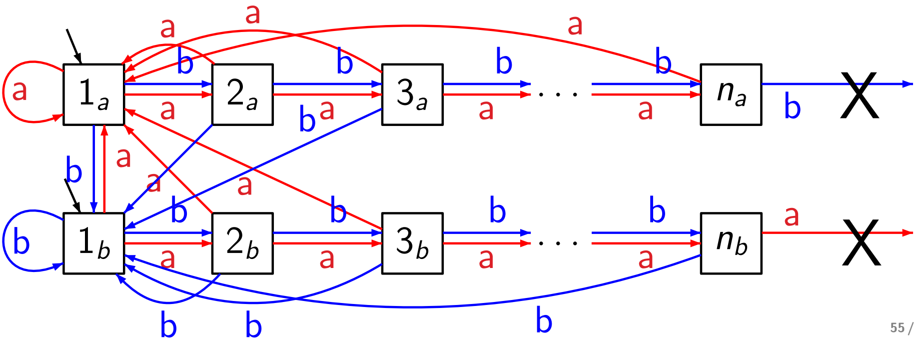
Efficiency of PBA

Thm: There exist uniform PBA $\mathcal{P}_n$ with $|\mathcal{P}_n| = \mathcal{O}(n)$
s.t. each equivalent NSA $\mathcal{A}_n$ has $\Omega(2^n/n)$ states.

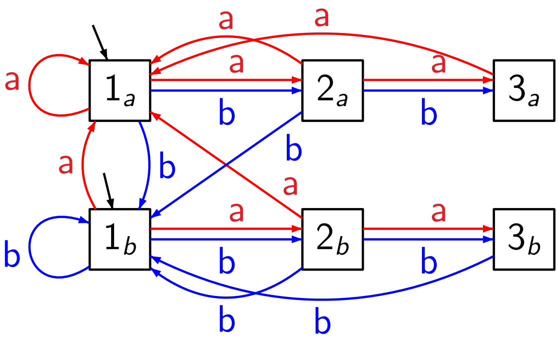
Proof: consider the ω-regular languages

$$L_n = \{ xy^\omega : x, y \in \{a, b\}^*, |y| = n \}$$

PBA for L_n mit $2n$ states:

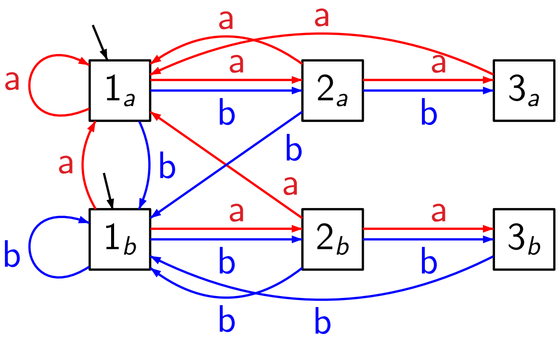


uPBA for $\mathsf{L}_3 = \{xy^\omega : x, y \in \{a, b\}^*, |y| = 3\}$



uniform distributions

uPBA for $L_3 = \{xy^\omega : x, y \in \{a, b\}^*, |y| = 3\}$

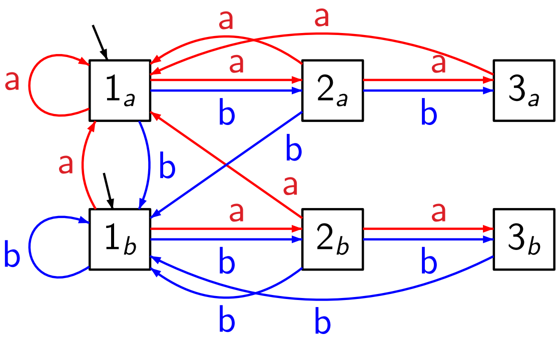


uniform distributions

1. Let $z = c_1 c_2 \ldots \in \{a, b\}^\omega$ s.t.

$$\overset{\infty}{\exists}\, i \text{ with } c_i = a \text{ and } c_{i+3} = b$$

uPBA for $L_3 = \{xy^\omega : x, y \in \{a, b\}^*, |y| = 3\}$



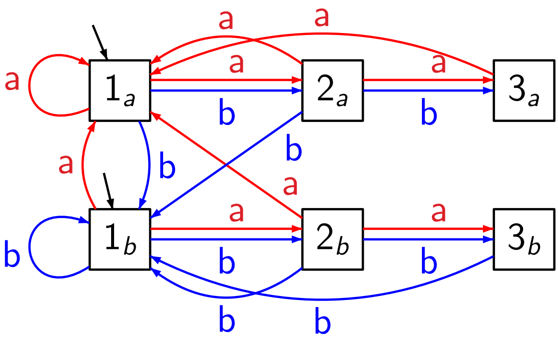
uniform distributions

1. Let $z = c_1 c_2 \ldots \in \{a, b\}^\omega$ s.t.

$$\overset{\infty}{\exists}\, i \text{ with } c_i = a \text{ and } c_{i+3} = b$$

with prob. 1: $\exists i$ s.t. $\xrightarrow{c_1 \ldots c_{i-1} a} 1_a \xrightarrow{c_{i+1}} 2_a \xrightarrow{c_{i+3}} = 3_a \overset{b}{\not\rightarrow}$

uPBA for $L_3 = \{xy^\omega : x, y \in \{a, b\}^*, |y| = 3\}$



uniform distributions

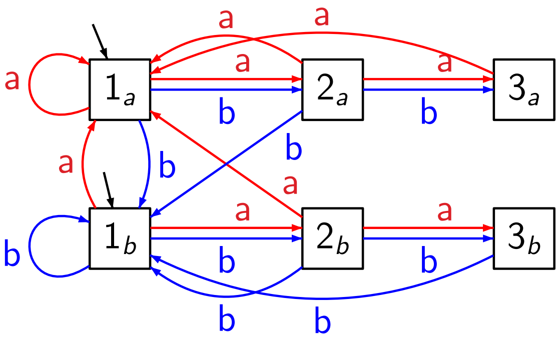
1. Let $z = c_1 c_2 \ldots \in \{a, b\}^\omega$ s.t.

$$\overset{\infty}{\exists}\, i \text{ with } c_i = a \text{ and } c_{i+3} = b$$

with prob. 1: $\exists i$ s.t. $\xrightarrow{c_1 \ldots c_{i-1} a} 1_a \xrightarrow{c_{i+1}} 2_a \xrightarrow{c_{i+3}} = 3_a \overset{b}{\not\to}$

i.e. z will be rejected almost surely

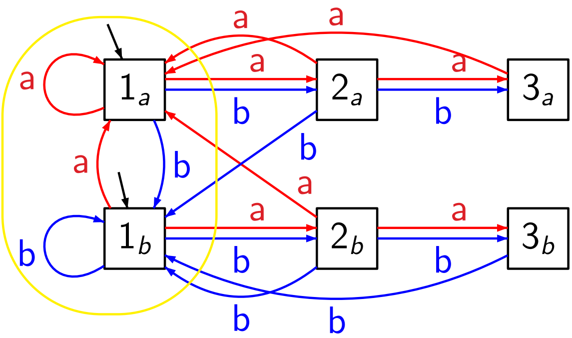
uPBA for $L_3 = \{xy^\omega : x, y \in \{a, b\}^*, |y| = 3\}$



uniform distributions

2. Let $z = xy^\omega \in L_3$ where $x = c_1 \ldots c_i$.

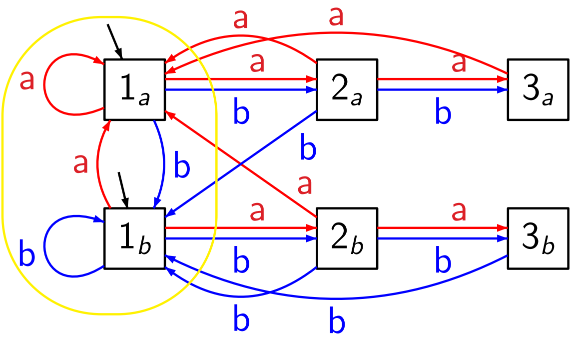
uPBA for $L_3 = \{xy^\omega : x, y \in \{a, b\}^*, |y| = 3\}$



uniform distributions

2. Let $z = xy^\omega \in L_3$ where $x = c_1 \dots c_i$.
   All runs for z that start with the prefix
   $1_a \xrightarrow{c_1} 1_{c_1} \xrightarrow{c_2} \dots \xrightarrow{c_i} 1_{c_i}$ are infinite

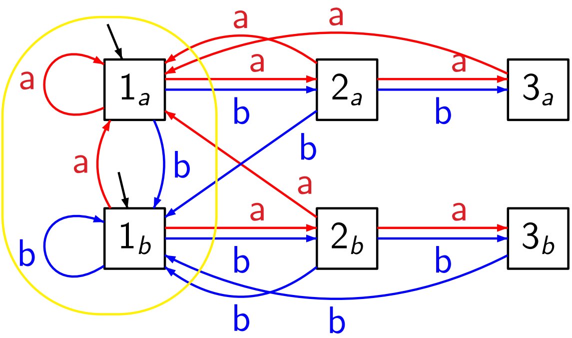
uPBA for $L_3 = \{xy^\omega : x, y \in \{a, b\}^*, |y| = 3\}$



uniform distributions

2. Let $z = xy^\omega \in L_3$ where $x = c_1 \ldots c_i$.
All runs for z that start with the prefix
$1_a \xrightarrow{c_1} 1_{c_1} \xrightarrow{c_2} \ldots \xrightarrow{c_i} 1_{c_i}$ are infinite (and accepting).

uPBA for $L_3 = \{xy^\omega : x, y \in \{a, b\}^*, |y| = 3\}$



uniform distributions

2. Let $z = xy^\omega \in L_3$ where $x = c_1 \ldots c_i$.
All runs for z that start with the prefix
$1_a \xrightarrow{c_1} 1_{c_1} \xrightarrow{c_2} \ldots \xrightarrow{c_i} 1_{c_i}$ are infinite (and accepting).
Hence: $\Pr(z) \geq (\frac{1}{2})^i > 0$

Efficiency of PBA

Thm: There exist uniform PBA with of size $\mathcal{O}(n)$ s.t. each equivalent NSA has $\Omega(2^n/n)$ states

Thm: There exist NBA $\mathcal{A}_n$ of size $\mathcal{O}(n)$ s.t. each equivalent uniform PBA has $\Omega(2^n)$ states

Efficiency of PBA

Thm: There exist NBA $\mathcal{A}_n$ of size $\mathcal{O}(n)$ s.t. each equivalent uniform PBA has $\Omega(2^n)$ states

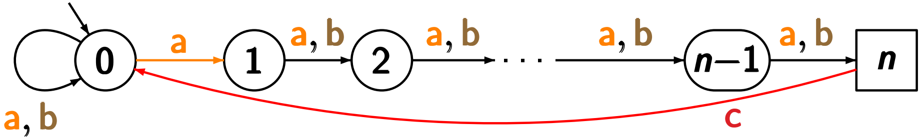
Proof: consider the language $((a+b)^* a (a+b)^{n-1} c)^\omega$

Efficiency of PBA

Thm: There exist NBA $\mathcal{A}_n$ of size $\mathcal{O}(n)$ s.t. each equivalent uniform PBA has $\Omega(2^n)$ states

Proof: consider the language $((a+b)^* a (a+b)^{n-1} c)^\omega$

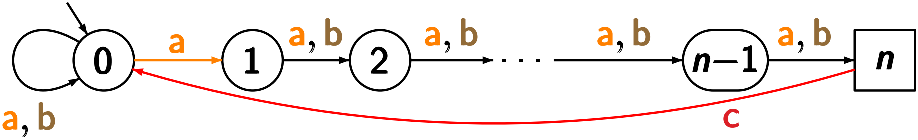
NBA:



Thm: There exist NBA $\mathcal{A}_n$ of size $\mathcal{O}(n)$ s.t. each equivalent uniform PBA has $\Omega(2^n)$ states

Proof: consider the language $((a+b)^* a (a+b)^{n-1} c)^\omega$

NBA:



PBA: has to "remember" all a's of the last n symbols

Probabilistic ω-automata

accepted language:

$$\mathcal{L}(\mathcal{P}) = \{ x \in \Sigma^\omega : Pr_{\mathcal{P}}(\text{accepting runs for } x)\} > 0 \}$$

acceptance conditions:

- Büchi: $\Box\Diamond F$
- Rabin: $\bigvee_{1 \leq i \leq k} (\Box\Diamond H_i \wedge \Diamond\Box \neg K_i)$
- Streett: $\bigwedge_{1 \leq i \leq k} (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

accepted language:

$$\mathcal{L}(\mathcal{P}) = \{x \in \Sigma^\omega : \Pr_{\mathcal{P}}(\text{ accepting runs for } x)\} > 0 \}$$

acceptance conditions:

- Büchi: $\Box\Diamond F$
- Rabin: $\bigvee_{1 \le i \le k} (\Box\Diamond H_i \wedge \Diamond\Box\neg K_i)$
- Streett: $\bigwedge_{1 \le i \le k} (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

PRA $\xrightarrow{\text{poly}}$ PBA as for nondeterministic automata

PSA $\xrightarrow{\text{exp}}$ PBA preserving uniformity

Probabilistic ω-automata

accepted language:

$$\mathcal{L}(\mathcal{P}) = \{\mathsf{x} \in \Sigma^\omega : \Pr_{\mathcal{P}}(\text{ accepting runs for } \mathsf{x})\} > 0 \}$$

acceptance conditions:

- Büchi: $\Box\Diamond F$
- Rabin: $\bigvee_{1 \le i \le k} (\Box\Diamond H_i \wedge \Diamond\Box\neg K_i)$
- Streett: $\bigwedge_{1 \le i \le k} (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

PRA $\xrightarrow{\text{poly}}$ PBA	as for nondeterministic automata
PSA $\xrightarrow{\text{exp}}$ PBA	preserving uniformity
PSA $\xrightarrow{\text{poly}}$ PBA	possibly non-uniform

Poly transformation from PSA to PBA

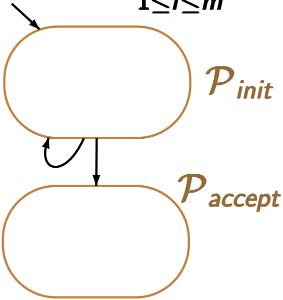
PSA $\mathcal{P}$ with acc. condition $\bigwedge_{1 \leq i \leq m} (\Box\Diamond \mathsf{H}_i \rightarrow \Box\Diamond \mathsf{K}_i)$

PBA consists of several copies of $\mathcal{P}$

Poly transformation from PSA to PBA

PSA $\mathcal{P}$ with acc. condition $\bigwedge_{1 \leq i \leq m} (\Box\Diamond H_i \to \Box\Diamond K_i)$

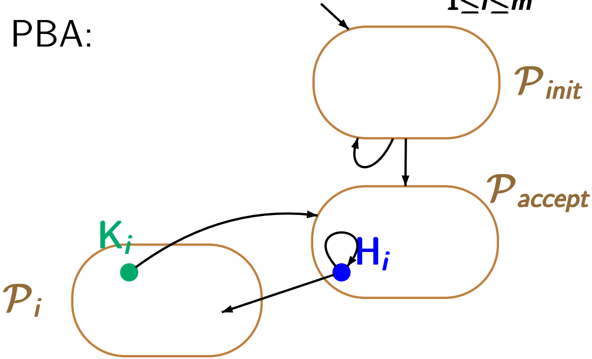
PBA:



Poly transformation from PSA to PBA

PSA $\mathcal{P}$ with acc. condition $\bigwedge_{1 \leq i \leq m} (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

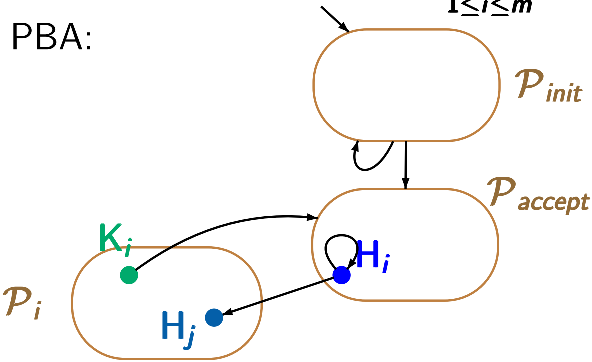
PBA:



Poly transformation from PSA to PBA

PSA $\mathcal{P}$ with acc. condition $\bigwedge_{1 \leq i \leq m} (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

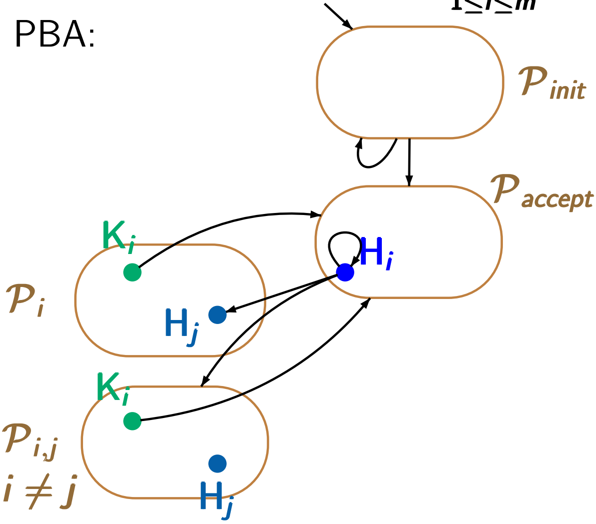
PBA:



Poly transformation from PSA to PBA

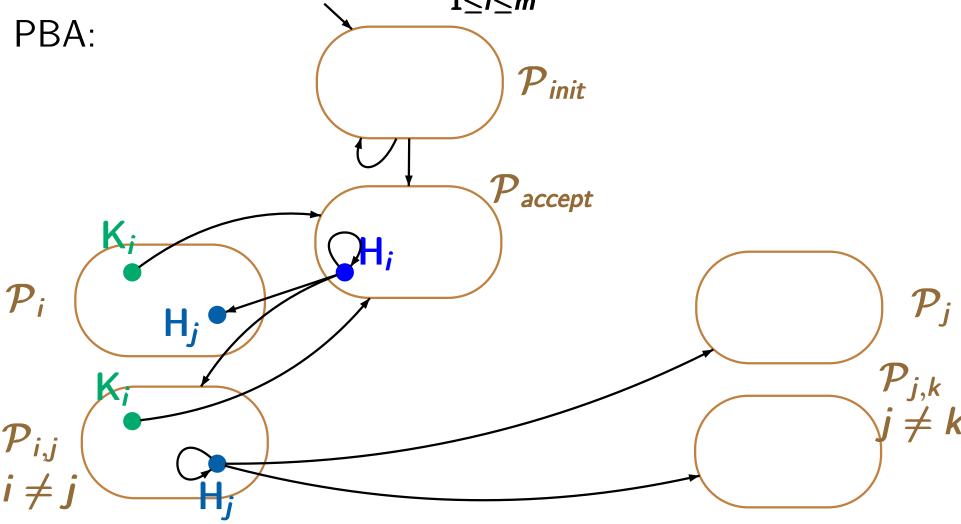
PSA $\mathcal{P}$ with acc. condition $\bigwedge_{1 \le i \le m} (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

PBA:



PSA $\mathcal{P}$ with acc. condition $\bigwedge_{1 \leq i \leq m} (\Box\Diamond H_i \to \Box\Diamond K_i)$

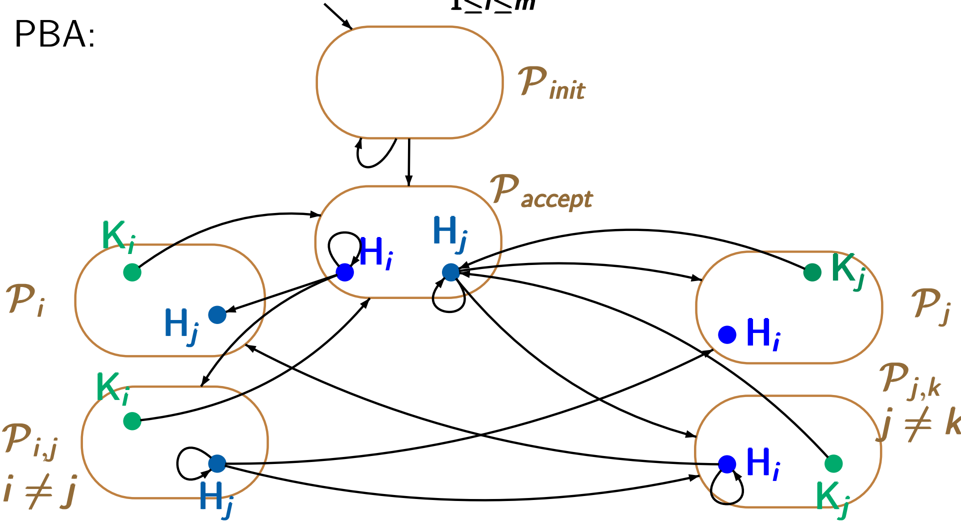
PBA:



Poly transformation from PSA to PBA

PSA $\mathcal{P}$ with acc. condition $\bigwedge_{1 \leq i \leq m} (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

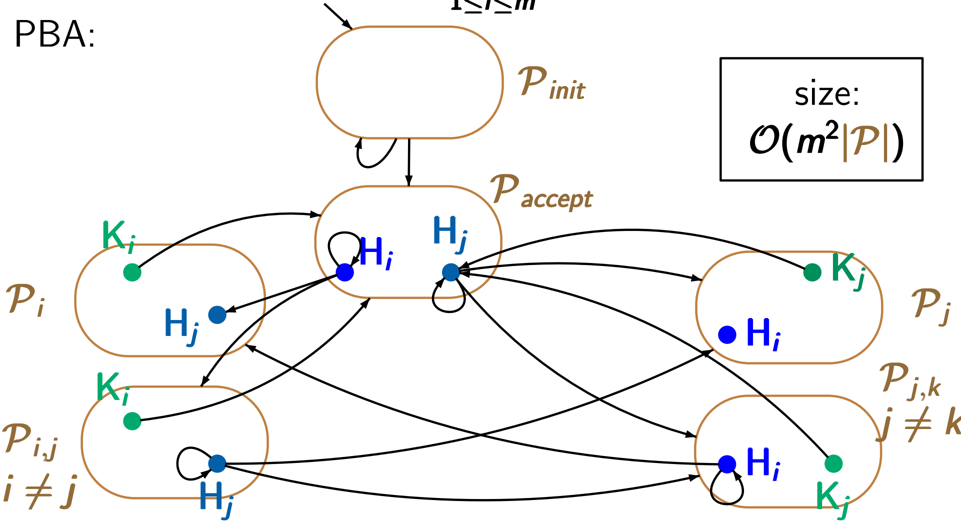
PBA:



Poly transformation from PSA to PBA

PSA $\mathcal{P}$ with acc. condition $\bigwedge_{1 \le i \le m} (\Box\Diamond H_i \rightarrow \Box\Diamond K_i)$

PBA:



size:
$\mathcal{O}(m^2|\mathcal{P}|)$

Alternative semantics for PBA

standard semantics:

$$\mathcal{L}(\mathcal{P}) \;=\; \{x \in \Sigma^\omega : \Pr_{\mathcal{P}}(x) > 0\}$$

Alternative semantics for PBA

standard semantics:

$$\mathcal{L}(\mathcal{P}) \;=\; \big\{ x \in \Sigma^\omega : \Pr_{\mathcal{P}}(x) > 0 \big\}$$

alternative semantics:

- threshold semantics
- almost-sure semantics

Threshold semantics for PBA

for PBA $\mathcal{P}$ and $\lambda \in]0,1[$

$$\mathcal{L}^{>\lambda}(\mathcal{P}) = \{ x \in \Sigma^{\omega} : \Pr_{\mathcal{P}}(x) > \lambda \}$$

Threshold semantics for PBA

for PBA $\mathcal{P}$ and $\lambda \in]0,1[$

$$\mathcal{L}^{>\lambda}(\mathcal{P}) \;=\; \{\mathsf{x} \in \mathbf{\Sigma}^{\boldsymbol{\omega}} : \mathsf{Pr}_{\mathcal{P}}(\mathsf{x}) > \lambda\}$$

Results:

- $\forall$ PBA $\mathcal{P}$ $\forall \lambda \in]0,1[$ $\exists$ PBA $\mathcal{P}'$ s.t. $\mathcal{L}^{>\lambda}(\mathcal{P}') = \mathcal{L}(\mathcal{P})$

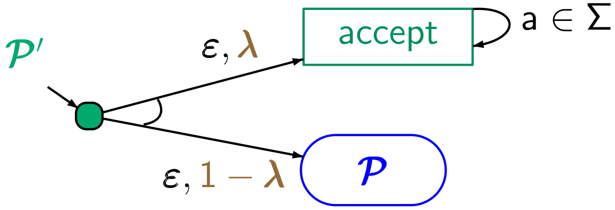
Threshold semantics for PBA

for PBA $\mathcal{P}$ and $\lambda \in]0,1[$

$$\mathcal{L}^{>\lambda}(\mathcal{P}) \;=\; \{x \in \Sigma^\omega : \Pr_{\mathcal{P}}(x) > \lambda\}$$

Results:

- $\forall$ PBA $\mathcal{P}$ $\forall \lambda \in]0,1[$ $\exists$ PBA $\mathcal{P}'$ s.t. $\mathcal{L}^{>\lambda}(\mathcal{P}') = \mathcal{L}(\mathcal{P})$



$\mathcal{P}'$ is uniform if $\mathcal{P}$ is uniform

Threshold semantics for PBA

for PBA $\mathcal{P}$ and $\lambda \in]0,1[$

$$\mathcal{L}^{>\lambda}(\mathcal{P}) \;=\; \big\{ \mathsf{x} \in \boldsymbol{\Sigma}^{\boldsymbol{\omega}} : \mathsf{Pr}_{\mathcal{P}}(\mathsf{x}) > \lambda \big\}$$

Results:

- $\forall$ PBA $\mathcal{P}$ $\forall \lambda \in]0,1[$ $\exists$ PBA $\mathcal{P}'$ s.t. $\mathcal{L}^{>\lambda}(\mathcal{P}') = \mathcal{L}(\mathcal{P})$
- $\exists$ PBA $\mathcal{P}$ $\exists \lambda \in]0,1[$ s.t. $\mathcal{L}^{>\lambda}(\mathcal{P})$ cannot be recognized by a standard PBA

Threshold semantics for PBA

for PBA $\mathcal{P}$ and $\lambda \in]0,1[$

$$\mathcal{L}^{>\lambda}(\mathcal{P}) \;=\; \{ x \in \Sigma^\omega : \Pr_{\mathcal{P}}(x) > \lambda \}$$

Results:

- $\forall$ PBA $\mathcal{P}$ $\forall \lambda \in]0,1[$ $\exists$ PBA $\mathcal{P}'$ s.t. $\mathcal{L}^{>\lambda}(\mathcal{P}') = \mathcal{L}(\mathcal{P})$
- $\exists$ PBA $\mathcal{P}$ $\exists \lambda \in]0,1[$ s.t. $\mathcal{L}^{>\lambda}(\mathcal{P})$ cannot be recognized by a standard PBA
- $\exists$ uniform PBA $\mathcal{P}$ $\exists \lambda \in]0,1[$ s.t $\mathcal{L}^{>\lambda}(\mathcal{P})$ is not ω-regular

Almost-sure semantics of PBA

$$\mathcal{L}^{=1}(\mathcal{P}) \;=\; \left\{ x \in \Sigma^\omega : \Pr(x) = 1 \right\}$$

Almost-sure semantics of PBA

$$\mathcal{L}^{=1}(\mathcal{P}) \; = \; \{ \mathsf{x} \in \Sigma^\omega : \mathsf{Pr}(\mathsf{x}) = 1 \}$$

Theorem: For each PBA $\mathcal{P}$ there exists a PBA $\mathcal{P}'$ such that $\mathcal{L}^{=1}(\mathcal{P}) = \mathcal{L}(\mathcal{P}')$.

Almost-sure semantics of PBA

$$\mathcal{L}^{=1}(\mathcal{P}) \;=\; \{x \in \Sigma^\omega : \Pr(x) = 1\}$$

Theorem: For each PBA $\mathcal{P}$ there exists a PBA $\mathcal{P}'$ such that $\mathcal{L}^{=1}(\mathcal{P}) = \mathcal{L}(\mathcal{P}')$.

Proof sketch. PBA $\mathcal{P}'$ for input word **x**

- simulates $\mathcal{P}$ with input **x**
- guesses at random a word position i
- checks whether $\Box\neg F$ holds with positive probability from position i
- if so, $\mathcal{P}'$ rejects; otherwise $\mathcal{P}'$ accepts.

Almost-sure semantics of PBA

$$\mathcal{L}^{=1}(\mathcal{P}) \;=\; \big\{ \mathsf{x} \in \Sigma^\omega : \Pr(\mathsf{x}) = 1 \big\}$$

Theorem: For each PBA $\mathcal{P}$ there exists a PBA $\mathcal{P}'$ such that $\mathcal{L}^{=1}(\mathcal{P}) = \mathcal{L}(\mathcal{P}')$.

Theorem: There exists a PBA $\mathcal{P}$ such that $\mathcal{L}(\mathcal{P})$ is *not* recognizable by a PBA with the almost-sure semantics.

Almost-sure semantics of PBA

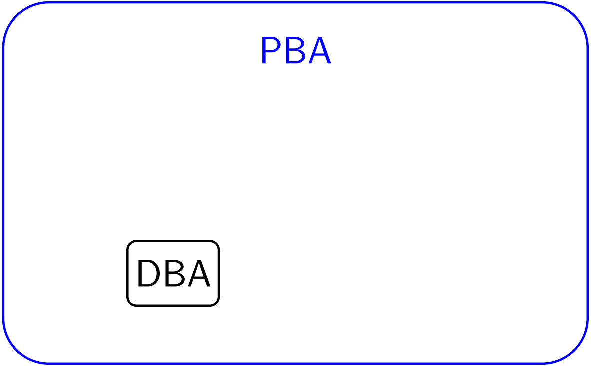
$$\mathcal{L}^{=1}(\mathcal{P}) = \{x \in \Sigma^\omega : \Pr(x) = 1\}$$

Theorem: For each PBA $\mathcal{P}$ there exists a PBA $\mathcal{P}'$ such that $\mathcal{L}^{=1}(\mathcal{P}) = \mathcal{L}(\mathcal{P}')$.

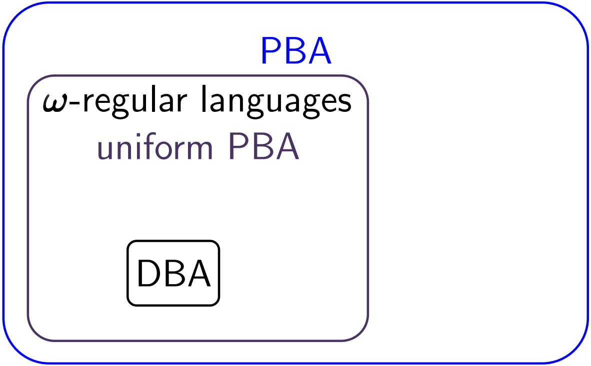
Theorem: There exists a PBA $\mathcal{P}$ such that $\mathcal{L}(\mathcal{P})$ is *not* recognizable by a PBA with the almost-sure semantics.

example: PBA for the ω-regular language $(a + b)^* a^\omega$

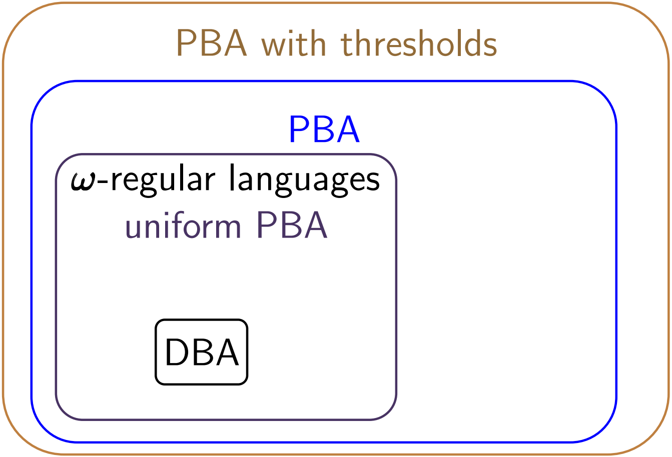
PBA with standard and alternative semantics

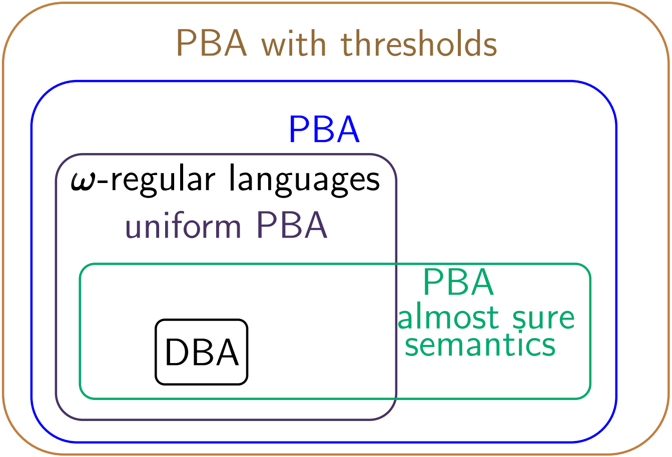


PBA with standard and alternative semantics

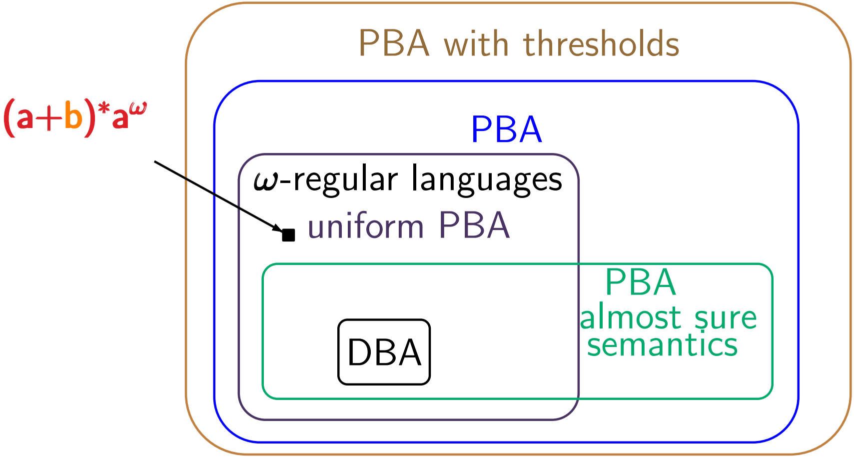


PBA with standard and alternative semantics



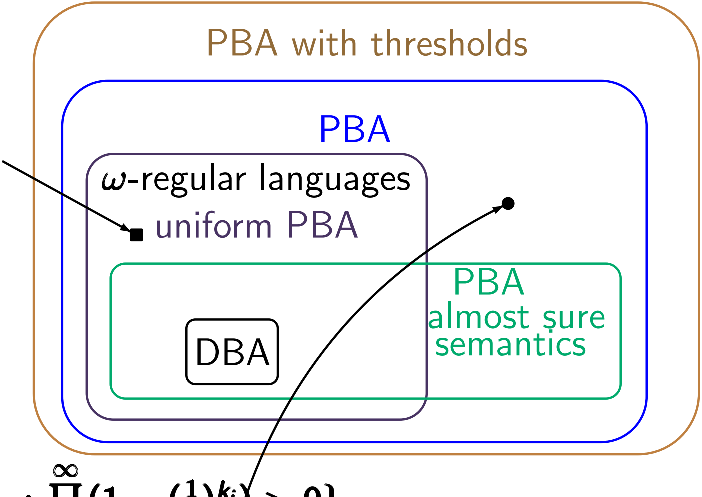

PBA with thresholds
PBA
ω-regular languages
uniform PBA
PBA
almost sure
semantics
DBA

PBA with standard and alternative semantics



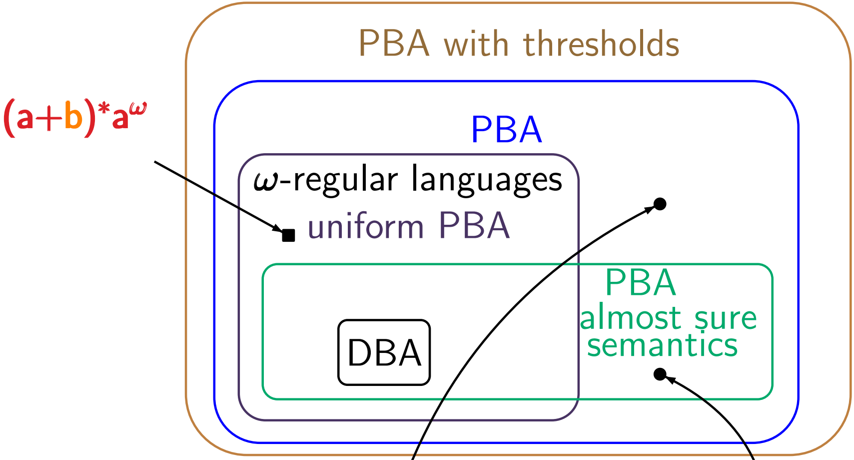
PBA with standard and alternative semantics

$(a+b)^*a^\omega$



$$\{a^{k_1}ba^{k_2}b\ldots : \prod_{i=1}^{\infty}(1-(\tfrac{1}{2})^{k_i}) > 0\}$$

PBA with standard and alternative semantics

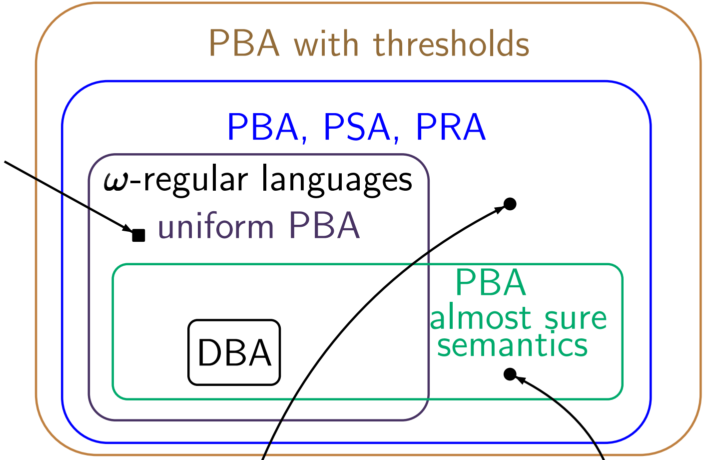


$$\{a^{k_1}ba^{k_2}b\ldots : \prod_{i=1}^{\infty}(1-(\tfrac{1}{2})^{k_i}) > 0\}$$

$$\{a^{k_1}ba^{k_2}b\ldots : \prod_{i=1}^{\infty}(1-(\tfrac{1}{2})^{k_i}) = 0\}$$

PBA with standard and alternative semantics

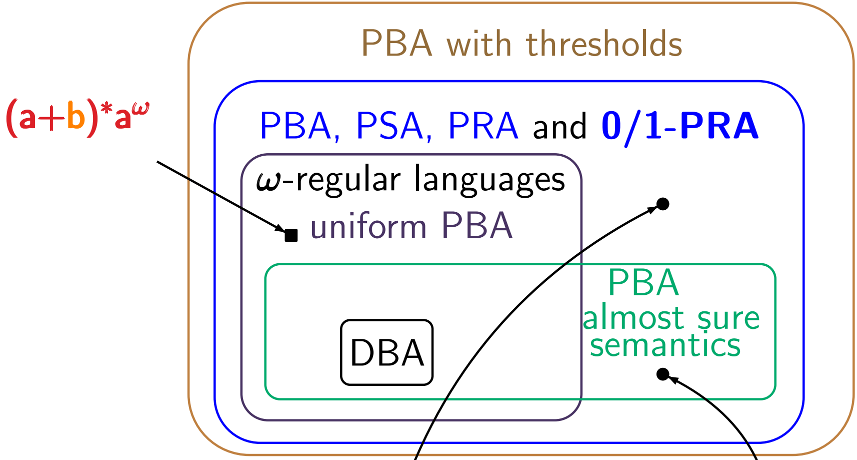
$(a+b)^*a^\omega$



$$\{a^{k_1}ba^{k_2}b\ldots : \prod_{i=1}^{\infty}(1-(\tfrac{1}{2})^{k_i}) > 0\}$$

$$\{a^{k_1}ba^{k_2}b\ldots : \prod_{i=1}^{\infty}(1-(\tfrac{1}{2})^{k_i}) = 0\}$$

PBA with standard and alternative semantics



$$\{a^{k_1}ba^{k_2}b\ldots : \prod_{i=1}^{\infty}(1-(\tfrac{1}{2})^{k_i}) > 0\}$$

$$\{a^{k_1}ba^{k_2}b\ldots : \prod_{i=1}^{\infty}(1-(\tfrac{1}{2})^{k_i}) = 0\}$$

0/1-PRA: probabilistic Rabin automaton $\mathcal{P}$ s.t.

$$\forall x \in \Sigma^\omega : \ \Pr_{\mathcal{P}}(x) \in \{0, 1\}$$

0/1-PRA

0/1-PRA: probabilistic Rabin automaton $\mathcal{P}$ s.t.

$$\forall x \in \Sigma^\omega : \ \Pr_{\mathcal{P}}(x) \in \{0, 1\}$$

Theorem: For each (standard) PBA $\mathcal{P}$ there exists a 0/1-PRA $\mathcal{P}_R$ such that $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{P}_R)$.

0/1-PRA

0/1-PRA: probabilistic Rabin automaton $\mathcal{P}$ s.t.

$$\forall x \in \Sigma^\omega : \ \Pr_{\mathcal{P}}(x) \in \{0,1\}$$

Theorem: For each (standard) PBA $\mathcal{P}$ there exists a 0/1-PRA $\mathcal{P}_R$ such that $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{P}_R)$.

Corollary: The almost-sure semantics for PRA is as powerful as the standard semantics.

The same holds for PSA, but not for PBA.

0/1-PRA: probabilistic Rabin automaton $\mathcal{P}$ s.t.

$$\forall x \in \Sigma^\omega : \; \Pr_{\mathcal{P}}(x) \in \{0, 1\}$$

Theorem: For each (standard) PBA $\mathcal{P}$ there exists a 0/1-PRA $\mathcal{P}_R$ such that $\mathcal{L}(\mathcal{P}) = \mathcal{L}(\mathcal{P}_R)$.

idea: 0/1-PRA $\mathcal{P}_R$

- generates up to $n = |Q|$ sample runs of $\mathcal{P}$ (as representatives for all runs in $\mathcal{P}$)
- and checks whether at least one of them is accepting

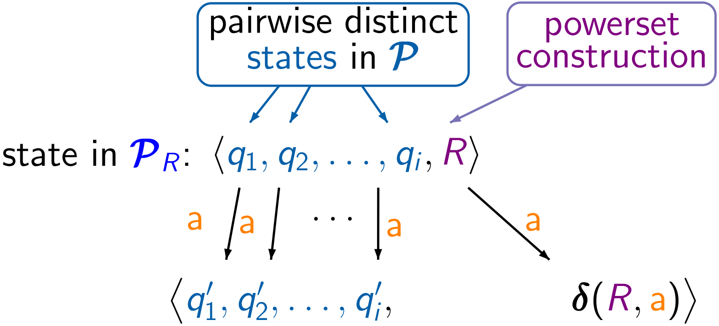
From PBA $\mathcal{P}$ to 0/1-PRA $\mathcal{P}_R$

pairwise distinct states in $\mathcal{P}$

powerset construction

state in $\mathcal{P}_R$: $\langle q_1, q_2, \ldots, q_i, R \rangle$

From PBA $\mathcal{P}$ to 0/1-PRA $\mathcal{P}_R$



δ = transition function in $\mathcal{P}$

From PBA $\mathcal{P}$ to 0/1-PRA $\mathcal{P}_R$

pairwise distinct states in $\mathcal{P}$

powerset construction

state in $\mathcal{P}_R$: $\langle q_1, q_2, \ldots, q_i, R \rangle$

$\downarrow a \quad \downarrow a \quad \cdots \quad \downarrow a \qquad \searrow a$

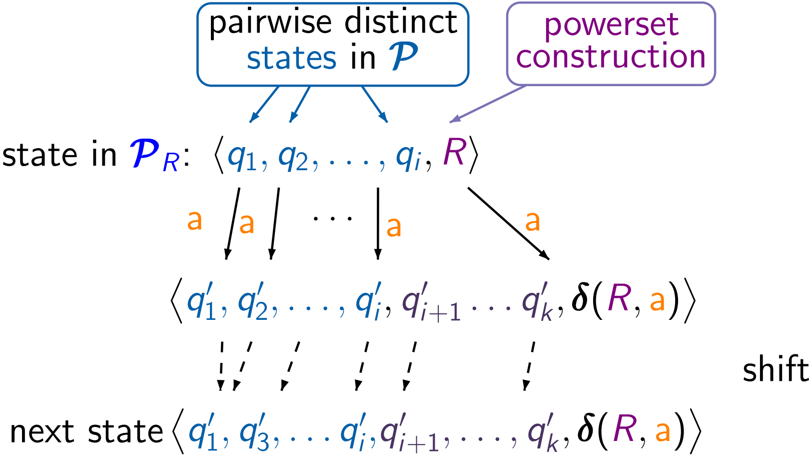
$$\left\langle q_1', q_2', \ldots, q_i', q_{i+1}' \ldots q_k', \boldsymbol{\delta}(R, a) \right\rangle$$

where $\{q_{i+1}', \ldots, q_k'\} = \mathbf{F} \cap \boldsymbol{\delta}(R, a) \setminus \{q_1', \ldots, q_i'\}$

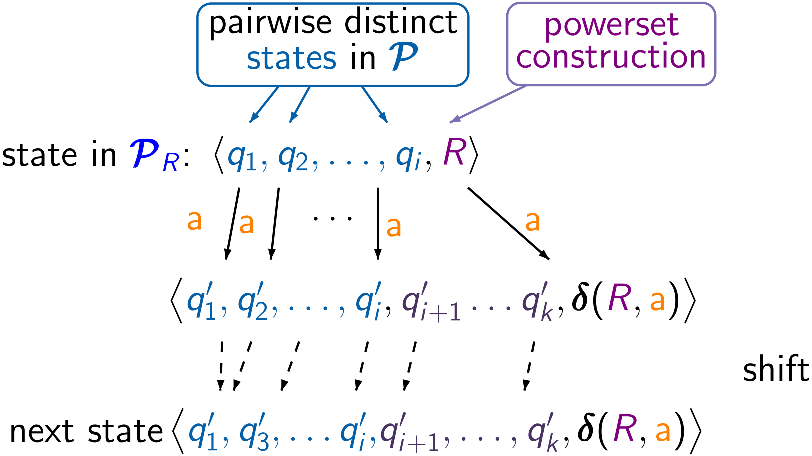
$\mathbf{F}$ = set of accept states in $\mathcal{P}$

$\boldsymbol{\delta}$ = transition function in $\mathcal{P}$

From PBA $\mathcal{P}$ to 0/1-PRA $\mathcal{P}_R$



From PBA $\mathcal{P}$ to 0/1-PRA $\mathcal{P}_R$



acceptance condition:

$$\bigvee_j (\Diamond\Box \text{“no shift in component j”} \wedge \Box\Diamond \mathbf{F})$$

Complementation of PBA

Theorem: The class of PBA-recognizable languages is closed under complementation.

Complementation of PBA

Theorem: The class of PBA-recognizable languages is closed under complementation.

PBA

$\mathcal{P}$

Theorem: The class of PBA-recognizable languages is closed under complementation.

PBA $\longrightarrow$ 0/1-PRA

$\mathcal{P}$ $\mathcal{P}_R$

Complementation of PBA

Theorem: The class of PBA-recognizable languages is closed under complementation.

$$\begin{array}{ccccc} \text{PBA} & \longrightarrow & \text{0/1-PRA} & \xrightarrow{\textbf{compl.}} & \text{0/1-PSA} \\ \mathcal{P} & & \mathcal{P}_R & & \end{array}$$

Complementation of PBA

Theorem: The class of PBA-recognizable languages is closed under complementation.

$$\begin{array}{ccccccc} \text{PBA} & \longrightarrow & 0/1\text{-PRA} & \xrightarrow{\textbf{compl.}} & 0/1\text{-PSA} \\ \mathcal{P} & & \mathcal{P}_R & \widehat{=} & \mathcal{P}_S \end{array}$$

Complementation of PBA

Theorem: The class of PBA-recognizable languages is closed under complementation.

$$\begin{array}{ccccccc} \text{PBA} & \longrightarrow & 0/1\text{-PRA} & \xrightarrow{\textbf{compl.}} & 0/1\text{-PSA} & \longrightarrow & \text{PBA} \\ \mathcal{P} & & \mathcal{P}_R & \widehat{=} & \mathcal{P}_S & & \overline{\mathcal{P}} \end{array}$$

Complementation of PBA

Theorem: The class of PBA-recognizable languages is closed under complementation.

$$\begin{array}{ccccccc} \text{PBA} & \xrightarrow{\textbf{exp}} & 0/1\text{-PRA} & \xrightarrow{\textbf{compl.}} & 0/1\text{-PSA} & \xrightarrow{\textbf{poly}} & \text{PBA} \\ \mathcal{P} & & \mathcal{P}_R & \widehat{=} & \mathcal{P}_S & & \overline{\mathcal{P}} \end{array}$$

Theorem: The class of PBA-recognizable languages is closed under complementation, union, intersection.

complementation:

$$\text{PBA} \xrightarrow{\textbf{exp}} 0/1\text{-PRA} \xrightarrow{\textbf{compl.}} 0/1\text{-PSA} \xrightarrow{\textbf{poly}} \text{PBA}$$

union and intersection:

- **union**: random choice between two PBA
- **intersection**: via generalized PBA (as for NBA)

Decision problems for PBA

The **emptiness problem** for PBA

given: PBA $\mathcal{P}$

question: does $\mathcal{L}(\mathcal{P}) = \emptyset$ hold?

Decision problems for PBA

The **emptiness problem** for PBA

given: PBA $\mathcal{P}$

question: does $\mathcal{L}(\mathcal{P}) = \emptyset$ hold?

is undecidable.

The **emptiness problem** for PBA

given: PBA $\mathcal{P}$

question: does $\mathcal{L}(\mathcal{P}) = \emptyset$ hold?

is undecidable.

proof by a reduction from the emptiness problem for **probabilistic finite automata (PFA)**

[Paz'71, Madani/Hanks/Condon'03]

Decision problems for PBA

The emptiness problem for PBA is undecidable.

Hence, the following problems are undecidable too:

Decision problems for PBA

The emptiness problem for PBA is undecidable.

Hence, the following problems are undecidable too:

- **universality**: given a PBA $\mathcal{P}$,
  does $\mathcal{L}(\mathcal{P}) = \Sigma^\omega$ hold?

The emptiness problem for PBA is undecidable.

Hence, the following problems are undecidable too:

- **universality**: given a PBA $\mathcal{P}$,
  does $\mathcal{L}(\mathcal{P}) = \Sigma^\omega$ hold?
- **equivalence**: given two PBA $\mathcal{P}_1, \mathcal{P}_2$,
  does $\mathcal{L}(\mathcal{P}_1) = \mathcal{L}(\mathcal{P}_2)$ hold?

Decision problems for PBA

The emptiness problem for PBA is undecidable.

Hence, the following problems are undecidable too:

- universality: does $\mathcal{L}(\mathcal{P}) = \Sigma^\omega$ hold?
- equivalence: does $\mathcal{L}(\mathcal{P}_1) = \mathcal{L}(\mathcal{P}_2)$ hold?
- **model checking** finite transition systems against PBA-specifications

Decision problems for PBA

The emptiness problem for PBA is undecidable.

Hence, the following problems are undecidable too:

- universality: does $\mathcal{L}(\mathcal{P}) = \Sigma^\omega$ hold?
- equivalence: does $\mathcal{L}(\mathcal{P}_1) = \mathcal{L}(\mathcal{P}_2)$ hold?
- **model checking** finite transition systems against PBA-specifications

  given a finite TS $\mathcal{T}$ and PBA $\mathcal{P}$, *does there exist* a path π in $\mathcal{T}$ s.t. $\mathit{trace}(\pi) \in \mathcal{L}(\mathcal{P})$?

Decision problems for PBA

The emptiness problem for PBA is undecidable.

Hence, the following problems are undecidable too:

- universality: does $\mathcal{L}(\mathcal{P}) = \Sigma^\omega$ hold?
- equivalence: does $\mathcal{L}(\mathcal{P}_1) = \mathcal{L}(\mathcal{P}_2)$ hold?
- **model checking** finite transition systems against PBA-specifications

  given a finite TS $\mathcal{T}$ and PBA $\mathcal{P}$, *does there exist* a path π in $\mathcal{T}$ s.t. $\mathit{trace}(\pi) \in \mathcal{L}(\mathcal{P})$?

  given a finite TS $\mathcal{T}$ and PBA $\mathcal{P}$, *does* $\mathit{trace}(\pi) \in \mathcal{L}(\mathcal{P})$ hold *for all* paths π in $\mathcal{T}$?

Decision problems for PBA

The following problems are undecidable:

- emptiness for PBA
- universality for PBA
- equivalence for PBA
- model checking finite TS against PBA-specifications
- verification of observation-based stochastic games formalized by **partially-observable MDPs (POMDPs)** against ω-regular specifications:

Decision problems for PBA

The following problems are undecidable:

- emptiness for PBA
- universality for PBA
- equivalence for PBA
- model checking finite TS against PBA-specifications
- verification of observation-based stochastic games formalized by **partially-observable MDPs (POMDPs)** against ω-regular specifications:
  - * does there exist a strategy **S** s.t. $\Pr_{\mathfrak{S}}(\Box\Diamond F) > 0$?
  - * does there exist a strategy **S** s.t. $\Pr_{\mathfrak{S}}(\Diamond\Box F) = 1$?

Decision problems for POMDPs

Theorem: The following instances of the verification problem for finite POMDPs against ω-regular specifications is **undecidable**:

* does there exist a strategy **S** s.t. $\Pr_{\mathfrak{S}}(\Box\Diamond F) > 0$?
* does there exist a strategy **S** s.t. $\Pr_{\mathfrak{S}}(\Diamond\Box F) = 1$?

Decision problems for POMDPs

Theorem: The following instances of the verification problem for finite POMDPs against ω-regular specifications is **undecidable**:

* does there exist a strategy **S** s.t. $\Pr_{\mathfrak{S}}(\Box\Diamond F) > 0$?
* does there exist a strategy **S** s.t. $\Pr_{\mathfrak{S}}(\Diamond\Box F) = 1$?

but the following problems are **decidable**:

* does there exist a strategy **S** s.t. $\Pr_{\mathfrak{S}}(\Box F) > 0$? *[de Alfaro'99]*

Decision problems for POMDPs

Theorem: The following instances of the verification problem for finite POMDPs against ω-regular specifications is **undecidable**:

* does there exist a strategy **S** s.t. $\mathrm{Pr}_{\mathsf{S}}(\Box\Diamond F) > \mathbf{0}$?
* does there exist a strategy **S** s.t. $\mathrm{Pr}_{\mathsf{S}}(\Diamond\Box F) = \mathbf{1}$?

but the following problems are **decidable**:

* does there exist a strategy **S** s.t. $\mathrm{Pr}_{\mathsf{S}}(\Box F) > \mathbf{0}$? *[de Alfaro'99]*
* does there exist a strategy **S** s.t. $\mathrm{Pr}_{\mathsf{S}}(\Diamond F) = \mathbf{1}$?

Decision problems for POMDPs

Theorem: The following instances of the verification problem for finite POMDPs against ω-regular specifications is **undecidable**:

* does there exist a strategy $\mathfrak{S}$ s.t. $\Pr_{\mathfrak{S}}(\Box\Diamond F) > 0$?
* does there exist a strategy $\mathfrak{S}$ s.t. $\Pr_{\mathfrak{S}}(\Diamond\Box F) = 1$?

but the following problems are **decidable**:

* does there exist a strategy $\mathfrak{S}$ s.t. $\Pr_{\mathfrak{S}}(\Box F) > 0$?
* does there exist a strategy $\mathfrak{S}$ s.t. $\Pr_{\mathfrak{S}}(\Diamond F) = 1$?
* does there exist a strategy $\mathfrak{S}$ s.t. $\Pr_{\mathfrak{S}}(\Box\Diamond F) = 1$?
* does there exist a strategy $\mathfrak{S}$ s.t. $\Pr_{\mathfrak{S}}(\Diamond\Box F) > 0$?

Theorem: The following instances of the verification problem for finite **POMDPs** are **decidable**:

⋮

does there exist a strategy **S** s.t. $\Pr_{\mathfrak{S}}(\Box\Diamond F) = 1$?

⋮

Decision problems for POMDPs

Theorem: The following instances of the verification problem for finite **POMDPs** are **decidable**:

$\vdots$

does there exist a strategy **S** s.t. $\Pr_{\mathfrak{S}}(\Box\Diamond F) = 1$?

$\vdots$

Corollary: The **emptiness problem** for **PBA** with the **almost-sure semantics** is **decidable**.

... undecidability results for general PBA, but ...

- emptiness problem for PBA with the almost-sure semantics is in **EXPTIME**

... undecidability results for general PBA, but ...

- emptiness problem for PBA with the almost-sure semantics is in **EXPTIME**
- emptiness problem for uniform PBA is in **EXPTIME**

- emptiness problem for PBA with the almost-sure semantics is in **EXPTIME**
- emptiness problem for uniform PBA is in **EXPTIME**

  via uPBA $\xrightarrow{\textbf{exp}}$ NSA

- emptiness problem for PBA with the almost-sure semantics is in **EXPTIME**
- emptiness problem for uniform PBA is in **EXPTIME**
- model checking Markov chains against uPBA-spec

  given: finite Markov chain $\mathcal{M}$, uniform PBA $\mathcal{P}$
  question: does $\mathbf{Pr}_{\mathcal{M}}(\mathcal{L}(\mathcal{P})) > \mathbf{0}$ hold?

- emptiness problem for PBA with the almost-sure semantics is in **EXPTIME**
- emptiness problem for uniform PBA is in **EXPTIME**
- model checking Markov chains against uPBA-spec

  given: finite Markov chain $\mathcal{M}$, uniform PBA $\mathcal{P}$
  question: does $\mathbf{Pr}_{\mathcal{M}}(\mathcal{L}(\mathcal{P})) > \mathbf{0}$ hold?

where $\mathbf{Pr}_{\mathcal{M}}(\mathcal{L}(\mathcal{P}))$ denotes the probability measure of the set of paths π in $\mathcal{M}$ s.t. $\mathbf{trace}(\pi) \in \mathcal{L}(\mathcal{P})$

- emptiness problem for PBA with the almost-sure semantics is in **EXPTIME**
- emptiness problem for uniform PBA is in **EXPTIME**
- model checking Markov chains against uPBA-spec

given: finite Markov chain $\mathcal{M}$, uniform PBA $\mathcal{P}$
question: does $\Pr_{\mathcal{M}}(\mathcal{L}(\mathcal{P})) > 0$ hold?

is in **PTIME** as

$$\Pr_{\mathcal{M}}(\mathcal{L}(\mathcal{P})) > 0 \quad \text{iff} \quad \Pr_{\mathcal{M}\times\mathcal{P}}(\Box\Diamond F) > 0$$

- emptiness problem for PBA with the almost-sure semantics is in **EXPTIME**
- emptiness problem for uniform PBA is in **EXPTIME**
- model checking Markov chains against uPBA-spec

  given: finite Markov chain $\mathcal{M}$, uniform PBA $\mathcal{P}$
  question: does $\mathbf{Pr}_{\mathcal{M}}(\mathcal{L}(\mathcal{P})) > \mathbf{0}$ hold?

  is in **PTIME** as

$$\mathbf{Pr}_{\mathcal{M}}(\mathcal{L}(\mathcal{P})) > \mathbf{0} \quad \text{iff} \quad \mathbf{Pr}_{\mathcal{M}\times\mathcal{P}}(\Box\Diamond \mathbf{F}) > \mathbf{0}$$

$$\uparrow$$

analysis of the SCCs in the product Markov chain

Conclusion

- **expressiveness:** PBA are more expressive than NBA and almost-sure PBA,

Conclusion

- **expressiveness:** PBA are more expressive than NBA and almost-sure PBA, while uniform PBA have equal power than NBA

Conclusion

- **expressiveness:** PBA are more expressive than NBA and almost-sure PBA, while uniform PBA have equal power than NBA
- **efficiency:** PBA can be exp smaller than NSA

Conclusion

- **expressiveness:** PBA are more expressive than NBA and almost-sure PBA, while uniform PBA have equal power than NBA
- **efficiency:** PBA can be exp smaller than NSA
- polynomial transformation PSA $\rightsquigarrow$ PBA

Conclusion

- expressiveness: PBA are more expressive than NBA and almost-sure PBA, while uniform PBA have equal power than NBA
- efficiency: PBA can be exp smaller than NSA
- polynomial transformation PSA $\rightsquigarrow$ PBA
- **undecidability** results for PBA and POMDPs

Conclusion

- expressiveness: PBA are more expressive than NBA and almost-sure PBA, while uniform PBA have equal power than NBA
- efficiency: PBA can be exp smaller than NSA
- polynomial transformation PSA $\rightsquigarrow$ PBA
- **undecidability** results for PBA and POMDPs
- **decidability** results for almost-sure PBA and POMDPs

Conclusion

- expressiveness: PBA are more expressive than NBA and almost-sure PBA, while uniform PBA have equal power than NBA
- efficiency: PBA can be exp smaller than NSA
- polynomial transformation PSA $\rightsquigarrow$ PBA
- undecidability results for PBA and POMDPs
- decidability results for almost-sure PBA and POMDPs
- **application:** run-time verification (probabilistic monitoring) [Sistla et al]

Conclusion

- expressiveness ...
- efficiency ....
- polynomial transformation PSA $\rightsquigarrow$ PBA
- (un)decidability results for PBA and POMDPs
- application: run-time verification [Sistla et al]

many open problems:

- transformations **LTL** $\rightsquigarrow$ **PBA** or **MSO** $\rightsquigarrow$ **PBA**
- alternative semantics for **PBA**
- variants: **NPBA**, **QBA**, ...