Conjunctive grammars with restricted disjunction

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Context-free grammars with Boolean operations

$G = (\Sigma, N, P, S)$, where...

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$$A \to \alpha$$

"If w is generated by α, then w is generated by A".

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Boolean grammars (Okhotin, 2003)

$$A \to \alpha_1 \& \ldots \& \alpha_m \& \neg\beta_1 \& \ldots \& \neg\beta_n$$

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  - $L_G(A) = \{w \mid A \Longrightarrow^* w\}$

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Conjunctive grammar for $\{a^n b^n c^n \mid n \geqslant 0\}$:

$$\begin{array}{rcll} S & \to & AB\&DC & \{a^n b^n c^n \mid n \geqslant 0\} \\ A & \to & aA \mid \varepsilon & a^* \\ B & \to & bBc \mid \varepsilon & \{b^k c^k \mid k \geqslant 0\} \\ C & \to & cC \mid \varepsilon & c^* \\ D & \to & aDb \mid \varepsilon & \{a^\ell b^\ell \mid \ell \geqslant 0\} \end{array}$$

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- Boolean grammar for a toy programming language (Okhotin, 2005).

The unary case: $\Sigma = \{a\}$

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Example (Jeż, DLT 2007)

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A_1 &\to A_1A_3 \& A_2A_2 \mid a \\
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- Implications on equations over sets of numbers (Jeż, Okhotin, STACS 2008; ICALP 2008; STACS 2009).

Practical properties

Greater expressive power:

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Key good properties of context-free grammars retained:

- Generated languages are in $DTIME(n^3) \cap DSPACE(n)$.
- Extension of generalized LR parsing (Okhotin, 2005).
- Extension of recursive descent parsing (Okhotin, 2007).

Binary normal form
(Okhotin, 2000)

Conjunctive grammar with all rules of the form

$$A \to B_1 C_1 \& \dots \& B_n C_n$$
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$$S \to \varepsilon$$

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- Effective transformation.
- Cubic-time parsing (extension of CKY).

Restricting the disjunction

Example ($\{a^n b^n c^n \mid n \geqslant 0\}$)

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  - Undecidability results.

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- Expressible in conjunctive grammars.

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A &\to aSa \mid \varepsilon \\
B &\to bSb \mid \varepsilon \\
O &\to OOO \mid a \mid b
\end{aligned}
$$

$$(aSa \cup \{\varepsilon\})(bSb \cup \{\varepsilon\}) \cap \mathrm{Odd} = (\cancelto{\varnothing}{aSabSb} \cup aSa \cup bSb \cup \cancelto{\varnothing}{\{\varepsilon\}}) \cap \mathrm{Odd} = aSa \cup bSb$$

- Essential: $L(aSa), L(bSb) \subseteq \mathrm{Odd}$.

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Theorem

Every conjunctive grammar can be transformed to ONF.

Restricting the disjunction

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If L is conjunctive, there is a restricted grammar for $L \cap \mathrm{Odd}$.

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Every regular language has a restricted conjunctive grammar without ε.

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  - Award of $240 Canadian for solving the problem.