On Finite Bases for Weak Semantics: Failures versus Impossible Futures

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Introduction (I)

- Labeled transition system: a fundamental model of concurrent computation.

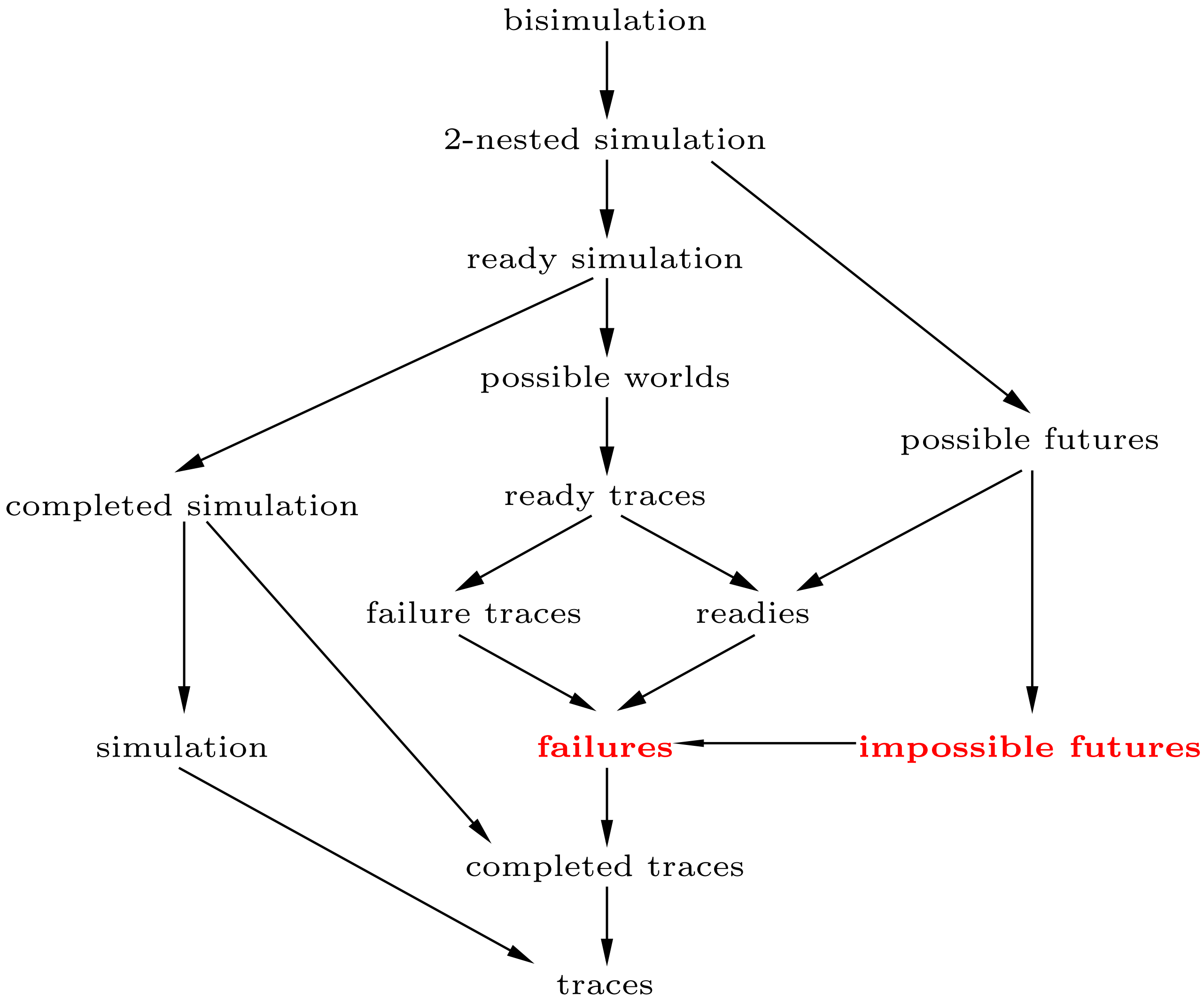
  $\mathcal{L} = (S, L, \{\xrightarrow{\ell}\}_{\ell \in L})$

  A labeled transition system contains a set of states, with typical element s, and a set of transitions $s \xrightarrow{\ell} s'$, where ℓ ranges over some set L of labels.

Introduction (II)

- Behavioral equivalence: identify those states of labeled transition systems that afford the same behaviors;
- No consensus on what is an appropriate notion of "behavioral equivalence" for reactive systems;
- One of the main tasks of concurrency theory is to provide a uniform classification of system (or process) behavior. This forms the study of comparative concurrency semantics;
- E.g. Linear time - branching time spectrum I of behavioral preorders and equivalences for finitely branching, sequential, concrete processes (van Glabbeek).

Linear Time - Branching Time Spectrum



Weak Semantics

Van Glabbeek's spectrum has two parts:

- Spectrum I deals with concrete processes (without silent moves τ), as showed before;

  $\longrightarrow$ concrete semantics

- Spectrum II expands I with one dimension to deal with processes with abstraction (silent moves τ)

  $\longrightarrow$ weak semantics;

Our work focuses on weak semantics. In particular, the axiomatizability of weak failures semantics and weak impossible futures semantics.

Failures and Impossible Futures

Assume a labeled transition system with the label set $A_\tau = A \cup \{\tau\}$:

- A pair $(a_1 \cdots a_k, B)$, with $k \geq 0$ and $B \subseteq A$, is a weak failure pair of a state s if there is a path $s \Rightarrow\xrightarrow{a_1}\Rightarrow \cdots \Rightarrow\xrightarrow{a_k}\Rightarrow s_k$ with $\mathcal{I}(s_k) \cap B = \emptyset$.

- A pair $(a_1 \cdots a_k, B)$, with $k \geq 0$ and $B \subseteq A^*$, is a weak impossible future of a state s if there is a path $s \Rightarrow\xrightarrow{a_1}\Rightarrow \cdots \Rightarrow\xrightarrow{a_k}\Rightarrow s_k$ with $\mathcal{T}(s_k) \cap B = \emptyset$.

Tip: $\Rightarrow$ is the transitive closure of $\xrightarrow{\tau}$; $\mathcal{I}(s) = \{a \in A \mid t \Rightarrow\xrightarrow{a}\}$ and $\mathcal{T}(s)$ denotes the set of traces of s.

These definitions naturally induce corresponding preorders and equivalences.

Axiomatization (I)

Process Calculi:

We work in the setting of the process algebra BCCS (BCCSP extended by τ):

BCCS	nil $\mathbf{0}$	prefixing αt (with $\alpha \in A_\tau$)
	choice $t + u$	variables x

$$\frac{}{\alpha x \xrightarrow{\alpha} x} \qquad \frac{x \xrightarrow{\alpha} x'}{x + y \xrightarrow{\alpha} x'} \qquad \frac{y \xrightarrow{\alpha} y'}{x + y \xrightarrow{\alpha} y'}$$

Note: BCCSP $=$ BCCS - τ

Axomatization (II)

- Axiomatization (a set of (in)equations) to characterize the semantics in the spectrum. For example:

  Axiomatization for concrete bisimulation:

$$
\begin{array}{lrcl}
\text{A1} & x + y & \approx & y + x \\
\text{A2} & (x + y) + z & \approx & x + (y + z) \\
\text{A3} & x + x & \approx & x \\
\text{A4} & x + \mathbf{0} & \approx & x
\end{array}
$$

  (In)Equational logic (rules): reflexivity, (symmetry), transitivity, substitution, and closure under context.

$$
\frac{t \approx u}{\sigma(t) \approx \sigma(u)} \qquad \frac{t \approx u}{C[t] \approx C[u]}
$$

Soundness and Ground-Completeness

Given a congruence $\sim$ over BCCSP, give a sound and ground-complete axiomatization $\approx$ for BCCSP:

$$P \sim Q \;\underset{\text{g-comp}}{\overset{\text{sound}}{\rightleftharpoons}}\; \vdash P \approx Q$$

for closed terms P, Q. (Likewise for preorders $\precsim$.)

For example,

$$P \underline{\leftrightarrow} Q \;\Leftrightarrow\; \text{A1} - \text{A4} \vdash P \approx Q \text{ for closed terms } P, Q$$

Note: $\sim = \precsim \cap \precsim^{-1}$;

Congruence: closed under the context of the process algebra

ω-Completeness

An axiomatization E is ω-complete if:

$$E \vdash \sigma(t) \approx \sigma(u) \text{ for all closed substitutions } \sigma \quad \Rightarrow \quad E \vdash t \approx u$$

Note that t and u are open terms, i.e. terms with variables.

Notable examples of ω-incomplete axiomatizations:

- $\lambda K\beta\eta$-calculus
- equational theory of CCS: laws such as commutativity of parallelism,

  $$t||(u||v) \approx (t||u)||v$$

  is valid in the initial model but cannot be derived.

In universal algebra, an ω-complete axiomatization is referred to as a basis for the equational theory of the algebra it axiomatizes.

Existing Results on Ground-completeness

ground	$1 \leq \|A\| < \infty$	$\|A\| = \infty$
bisim	$+$	$+$
2-nested sim	$-$	$-$
possible futu	$-$	$-$
ready sim	$+$	$+$
compl sim	$+$	$+$
sim	$+$	$+$
possible worl	$+$	$+$
ready tr	$+$	$-$
readies	$+$	$+$
failure tr	$+$	$+$
failures	$+$	$+$
completed tr	$+$	$+$
partial tr	$+$	$+$

van Glabbeek

Aceto, Fokkink,
van Glabbeek, Ingolfsdottir

Blom, Fokkink, Nain

Existing Results on ω-completeness

	$\lvert A\rvert = 1$	$1 < \lvert A\rvert < \infty$	$\lvert A\rvert = \infty$
bisim	$+$	$+$	$+$
2-nes sim	$-$	$-$	$-$
poss futu	$-$	$-$	$-$
ready sim	$\oplus$	$-$	$+$
compl sim	$\oplus$	$-$	$-$
sim	$\oplus$	$-$	$+$
poss worl	$\oplus$	$-$	$+$
ready tr	$\oplus$	$-$	$-$
readies	$\oplus$	$-$	$+$
failure tr	$\oplus$	$-$	$+$
failures	$\oplus$	$\oplus$	$+$
compl tr	$\oplus$	$+$	$+$
partial tr	$\oplus$	$+$	$+$

Problems

We have achieved a (relatively) comprehensive understanding on the axiomatizability of concrete semantics. But

$\boxed{\text{What about weak semantics?}}$

Some existing results:

- Milner's classical axiomatization for observational congruence;
- Van Glabbeek's ground-complete axiomatizations for weak failures, weak traces, etc;
- [Voorhoeve and Mauw, 2001]: (1) an inequational axiomatization of BCCS modulo weak impossible futures preorder; (2) ω-completeness in case of an infinite alphabet.

This talk: Answers the questions regarding failures and impossible futures. For each one, we consider two dimensions: (1) preorder vs. equivalence; (2) ground-completeness vs. ω-completeness.

Congruences

Please bear in mind: the axiomatizability only makes sense when congruences are considered.

Failures:

- The weak failures preorder $\sqsubseteq_{\mathrm{WF}}$ is given by: $p \sqsubseteq_{\mathrm{WF}} q$ iff (1) the weak failure pairs of p are also weak failure pairs of q and (2) $p \xrightarrow{\tau}$ implies that $q \xrightarrow{\tau}$.

- Weak failures equivalence $\equiv_{\mathrm{WF}}$ is defined as $\sqsubseteq_{\mathrm{WF}} \cap \sqsubseteq_{\mathrm{WF}}^{-1}$.

Impossible Futures:

- The weak impossible futures preorder $\sqsubseteq_{\mathrm{WIF}}$ is given by: $p \sqsubseteq_{\mathrm{WIF}} q$ iff (1) the weak impossible futures of p are also weak impossible futures of q, (2) $\mathcal{T}(p) = \mathcal{T}(q)$ and (3) $p \xrightarrow{\tau}$ implies that $q \xrightarrow{\tau}$.

- Weak impossible futures equivalence $\equiv_{\mathrm{WIF}}$ is defined as $\sqsubseteq_{\mathrm{WIF}} \cap \sqsubseteq_{\mathrm{WIF}}^{-1}$.

Failures

Let us first recall some result regarding concrete semantics.

On BCCSP processes, a ground-complete axiomatization for (concrete) failures preorder exists. It consists of the core axioms A1-4 together with one extra axiom:

$$\mathrm{F} \qquad a(x+y) \preccurlyeq ax + a(y+z)$$

On BCCS processes, the weak failures preorder coincides with the inverse of the must-testing preorder due to De Nicola and Hennessy.

$$\begin{array}{llcl}
\mathsf{WF1} & ax + ay & \approx & a(\tau x + \tau y) \\
\mathsf{WF2} & \tau(x+y) & \preccurlyeq & \tau x + y \\
\mathsf{WF3} & x & \preccurlyeq & \tau x + y
\end{array}$$

ω-completeness?

We extend this ground-completeness result with two ω-completeness results.

(1) Infinite alphabet: Yes!

If $|A| = \infty$, then $\mathrm{A1\text{-}4{+}WF1\text{-}3}$ is ω-complete for $\mathrm{BCCS}(A)$ modulo $\sqsubseteq_{\mathrm{WF}}$.

(2) Finite alphabet: A bit more involved ...

To get a finite basis for the inequational theory of BCCS modulo $\sqsubseteq_{\mathrm{WF}}$ in case $|A| < \infty$, we need to add the following axiom:

$$\mathrm{WF}_A \qquad \sum_{a \in A} a x_a \preccurlyeq \sum_{a \in A} a x_a + y$$

where the x_a for $a \in A$ and y are distinct variables.

Equivalences: "Ready to Preorder" approach

For BCCSP, Aceto *et al* (see also Frutos Escrig *et al*) proposed an algorithm $\mathcal{A}$

an inequational axiomatization E for $\sqsubseteq$ $\longrightarrow$

an equational axiomatization $\mathcal{A}(E)$ for $\sqsubseteq \cap \sqsubseteq^{-1}$.

The axiomatization $\mathcal{A}(E)$ generated by the algorithm from E contains

- the axioms A1-4 for bisimulation equivalence and the axioms $b(ax + z) + b(ax + by + z) \approx b(ax + ay + z)$ for $a, b \in A$ (RS)
- for each inequational axiom $t \preccurlyeq u$ in E:
  - $t + u \approx u$; and
  - $a(t + x) + a(u + x) \approx a(u + x)$ (for each $a \in A$, and some variable x that does not occur in $t + u$).

ω-Complete Axiomatization for Equivalences

Recently, we lifted this result to weak semantics, with some extra technical conditions, which allows us to apply the algorithm to weak failures.

After simplification and omission of redundant axioms, we obtain the following axiomatization(s) for equivalences.

$$
\begin{array}{llcl}
\text{WF1} & ax + ay & \approx & a(\tau x + \tau y) \\
\text{WFE2} & \tau(x + y) + \tau x & \approx & \tau x + y \\
\text{WFE3} & ax + \tau(ay + z) & \approx & \tau(ax + ay + z) \\
\hline
\text{WFE}_A & \tau(\sum_{a \in A} ax_a + y + z) & \approx & \tau(\sum_{a \in A} ax_a + y + z) \\
 & & & \quad + \tau(\sum_{a \in A} ax_a + z)
\end{array}
$$

Quick Summary

Weak failures semantics enjoys very nice axiomatizability properties:

- There exists a finite, ground-complete axiomatization for weak failures preorders;
- The finite, ground-complete axiomatization for weak failures preorder is ω-complete in case of an infinite alphabet;
- We can obtain an ω-complete axiomatization in case of finite alphabet, by adding one extra axiom;
- All of above apply to weak failures equivalences.

In short: everything is possible! :-)

Impossible Futures

Let's first recall some result regarding concrete semantics.

[Chen and Fokkink, LICS'08]

- We find a finite, sound, ground-complete axiomatization for BCCSP modulo (concrete) impossible futures preorder $\precsim_{\mathrm{IF}}$.
- Unfortunately, there does not exist any finite, sound, ground-complete axiomatization for BCCSP modulo (concrete) impossible futures equivalence $\simeq_{\mathrm{IF}}$.
- There does not exists any finite, sound, ω-complete axiomatization when the alphabet is finite.

Weak Case?

A ground-complete axiomatization from concrete case

$$
\begin{array}{lrcl}
\text{IF1} & a(x+y) & \preccurlyeq & ax+ay \\
\text{IF2} & a(x+y)+ax+a(y+z) & \approx & ax+a(y+z)
\end{array}
$$

to weak case

$$
\begin{array}{lrcll}
\text{WIF1 (WF1)} & ax+ay & \approx & a(\tau x+\tau y) & \\
\text{WIF2 (WF2)} & \tau(x+y) & \preccurlyeq & \tau x+y & \\
\text{WIF3} & x & \preccurlyeq & \tau x & \{+y\}
\end{array}
$$

BTW: this is a simplification of the axiomatization given in [VM01].

Moreover, it is ω-complete in case of an infinite alphabet.

Equivalence

Claim: There does not exist any finite, sound, ground-complete axiomatization for BCCS modulo (weak) impossible futures equivalence $\equiv_{\mathrm{WIF}}$.

The cornerstone is the following infinite family of closed equations, for $m \geq 0$:

$$\tau a^{2m}\mathbf{0} + \tau(a^m\mathbf{0} + a^{2m}\mathbf{0}) \;\approx\; \tau(a^m\mathbf{0} + a^{2m}\mathbf{0})$$

It is not hard to see that they are sound modulo $\equiv_{\mathrm{WIF}}$. But it can be shown that any finite sound axiomatization cannot drive all of them.

How to Prove?

Proof theoretic approach:

Step 1: Find sound (in)equations e_n $(n \geq 1)$ (such that any finite sound axiomatization E does not prove all e_n).

Step 2: Give a property P_E of equations that:

- holds true for each instantiation of the axioms in E;
- is preserved by the rules of (in)equational logic; and
- fails for some e_n.

  $\Longrightarrow$ Contradiction!

Applying this technique

For $m \geq 0$:

$$\tau^{2m}\mathbf{0} + \tau(a^m\mathbf{0} + a^{2m}\mathbf{0}) \;\approx\; \tau(a^m\mathbf{0} + a^{2m}\mathbf{0})$$

Lemma: Assume that, for E an axiomatization sound for $\sqsubseteq_{\mathrm{WIF}}$, closed terms p, q, closed substitution σ, action a and integer m:

1. $E \vdash p \approx q$;
2. $m > \max\{|u| \mid t \approx u \in E\}$;
3. $\mathcal{CT}(q) \subseteq \{a^m, a^{2m}\}$; and
4. there is a closed term p' such that $p \Rightarrow\xrightarrow{\tau} p'$ and $\mathcal{CT}(p') = \{a^{2m}\}$.

Then there is a closed term q' such that $q \Rightarrow\xrightarrow{\tau} q'$ and $\mathcal{CT}(q') = \{a^{2m}\}$.

Remarks

- Impossible futures semantics is the first example that affords a ground-complete axiomatization for BCCS modulo the preorder, while missing a ground-complete axiomatization for BCCS modulo the equivalence.
- "Ready to preorder" algorithm? That algorithm only applies to semantics that are at least as coarse as ready simulation semantics. Since impossible futures semantics is incomparable to ready simulation semantics, it falls outside the scope of the algorithm.
- Main reason: $a(bx + bx + z) \approx a(bx + bx + z) + a(bx + z)$ does not hold anymore.

ω-Completeness for $\sqsubseteq_{\mathrm{WIF}}$

Results summary:

- Negative results on equivalence are inherited, since ω-completeness is stronger property.
- In case $|A| = \infty$, we prove that there exists a finite basis for the inequational theory of $\mathrm{BCCS}(A)$ modulo $\sqsubseteq_{\mathrm{WIF}}$. The proof is based on an adaptation of Groote's inverted substitutions technique to inequations.
- In case $|A| < \infty$, we prove that a finite basis does not exist. We give two different proofs of this last fact, one for the case $1 < |A| < \infty$ and one for the case $|A| = 1$.

Finite Alphabet (I)

We prove that, if $1 < |A| < \infty$, the inequational theory of $\mathrm{BCCS}(A)$ modulo $\sqsubseteq_{\mathrm{WIF}}$ does not have a finite basis.

The cornerstone for this negative result is the following infinite family of inequations, for $m \geq 0$:

$$\tau(a^m x) + \Phi_m \ \preccurlyeq \ \Phi_m$$

with

$$\Phi_m \ = \ \tau(a^m x + x) + \sum_{b \in A} \tau(a^m x + a^m b\mathbf{0})$$

It is not hard to see that these inequations are sound modulo $\sqsubseteq_{\mathrm{WIF}}$.

Finite Alphabet (II)

Also, the inequational theory of $\mathrm{BCCS}(A)$ modulo $\sqsubseteq_{\mathrm{WIF}}$ does not have a finite basis in case of a singleton alphabet.

The cornerstone for the negative result for $|A| = 1$ is the following infinite family of inequations, for $m \geq 0$:

$$a^m x \;\preccurlyeq\; a^m x + x$$

If $|A| = 1$, then these inequations are clearly sound modulo $\sqsubseteq_{\mathrm{WIF}}$. Note that given a closed substitution ρ, $\mathcal{T}(\rho(x)) \subseteq \mathcal{T}(\rho(a^m x))$.

Theorem: For $|A| < \infty$, the inequational theory of $\mathrm{BCCS}(A)$ modulo $\sqsubseteq_{\mathrm{WIF}}$ does not have a finite basis.

Conclusion

Failures versus Impossible Futures:

- Ground-completeness for preorder: $x \preccurlyeq \tau x + y$ versus $x \preccurlyeq \tau x$;
- Ground-completeness for equivalence: Yes versus No;
- ω-completeness (infinite alphabet): $x \preccurlyeq \tau x + y$ versus $x \preccurlyeq \tau x$;
- ω-completeness (finite alphabet): Yes versus No;

Future works: (1) More axiomatizability results for weak semantics? (2) Establish links between axiomatizabilities of concrete and weak semantics.

Thank you for your attention!